Two Approaches to Discrete Conformal Maps

Peter Schröder

Context

Discrete analytic functions

- conformal maps
  - parameterization
  - remapping of domains
- what are the right discretizations?
  - DEC derived approaches
  - circle packings

The DEC Way

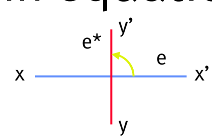
Mercat's approach

- discrete Riemann surfaces
- start with cellular decomposition
- add discrete metric: length of edge
  - need dual for analytic geometry
- discrete conformal structure
  - only primal/dual edge length relevant

Discrete Holomorphy

Discrete Cauchy-Riemann equations

$$\boxed{\frac{f(y')-f(y)}{l(y,y')} = i\frac{f(x')-f(x)}{l(x,x')}}$$



- Σ : oriented surface w. boundary
- Γ : discrete cellular decomposition
- $C_\bullet(\Gamma)$: chains, Γ_0, Γ_1, Γ_2
- $\partial : C_k(\Gamma) \to C_{k\text{-}1}(\Gamma)$

Setup Continued

- $\Lambda = \Gamma \cup \Gamma^*$: the double ← every $e \in \Gamma_1$ gets a $\diamond$
- conformally equivalent metrics

$$\boxed{\rho(e) = \frac{l(e^*)}{l(e)} = \frac{l'(e^*)}{l'(e)}}$$

- $C^k(\Lambda) := Hom(C_k(\Lambda), \mathbb{C})$
- pairings:

$$\begin{array}{lll} f(x) & f \in C^0(\Lambda) & x \in \Lambda_0 \\ \int_e \alpha & \alpha \in C^1(\Lambda) & e \in \Lambda_1 \\ \int_F \omega & \omega \in C^2(\Lambda) & F \in \Lambda_2 \end{array}$$

Co-Boundary & Hodge $*$

- $d: C^k(\Lambda) \to C^{k+1}(\Lambda)$

$$\int_{(x,x')} df = f(\partial(x,x')) = f(x') - f(x) \qquad \int_F d\alpha = \int_{\partial F} \alpha$$

- $* : C^k(\Lambda) \to C^{2-k}(\Lambda)$

$$\int_e *\alpha = -\rho(e^*) \int_{e^*} \alpha \qquad \int_F *f = f(F^*) \qquad *\omega(x) = \int_{x^*} \omega$$

- eigen spaces of $*$

$$\boxed{\rho(e)\rho(e^*) = \frac{l(e^*)}{l(e)} \frac{l(e)}{l(e^*)} = 1}$$

$$\boxed{*^2 = (-1)^{k(2-k)} \mathrm{Id}_{C^k(\Lambda)}} \qquad \boxed{C^1(\Lambda) = C^{(1,0)} \oplus C^{(0,1)}(\Lambda)}$$

Holomorphic 1-Forms

Projectors

$$P_{(1,0)} = \frac{1}{2}(\mathrm{Id} + i*) : C^1(\Lambda) \to C^{(1,0)}(\Lambda)$$

$$P_{(0,1)} = \frac{1}{2}(\mathrm{Id} - i*) : C^1(\Lambda) \to C^{(0,1)}(\Lambda)$$

holomorphic part

anti-holomorphic part

- closed and of type (1,0)

$$\boxed{\alpha \in \Omega^1(\Lambda) \Leftrightarrow d\alpha = 0 \quad \text{and} \quad *\alpha = -i\alpha}$$

- meromorphic with pole at $x \in \Lambda_0$

$$\mathrm{Res}_x(\alpha) = \frac{1}{2i\pi}\int_{\partial x^*} \alpha$$

Laplace's Equation

Our old friend

$$\Delta := -d*d* - *d*d$$

cot formula

$$\boxed{\Delta f(x) = \sum_{(x,x')\in\Lambda_1} \rho(e)(f(x) - f(x'))}$$

- in the compact case: $d^* = -*d*$

$$(f,g) = \sum_{x\in\Lambda_0} f(x)\bar{g}(x) \qquad (\alpha,\beta) = \sum_{e\in\Lambda_1} \rho(e)\int_e \alpha \int_e \bar{\beta} \qquad (\omega,\sigma) = \sum_{F\in\Lambda_2} \int_F \omega \int_F \bar{\sigma}$$

$$\boxed{C^k(\Lambda) = \mathrm{Im}\, d \oplus^{\perp} \mathrm{Im}\, d^* \oplus^{\perp} \mathrm{Ker}\, \Delta}$$

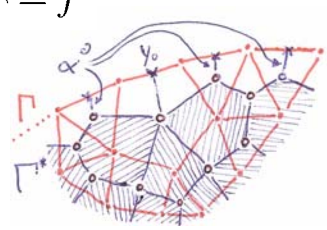
Laplace's Equation

Finite, connected Γ & $\rho:\Gamma_1 \to (0,\infty)$

- Dirichlet ($f^{\partial}:(\partial\Gamma)_0 \to \mathbb{C}$)

$$\Delta f = 0|_{\Gamma\setminus(\partial\Gamma)_0} \qquad f|_{\partial\Gamma} = f^{\partial}$$

- Neumann



$$\Delta f = 0|_{\Gamma^*\setminus(\partial\Gamma^*)_0}$$

$e_0 \neq e \in (\partial\Gamma^*)_1$

$$f(y_0) = f_0 \qquad \int_e df = \int_e \alpha$$

- dual problem: $i{*}\alpha$, integrate $i{*}df$

Computationally…

Split onto Γ and Γ^*

- boundary, co-boundary, Laplacian
- conformal map is conjugate pair

$$\boxed{\alpha_\Gamma \qquad \alpha_{\Gamma^*} := i * \alpha_\Gamma}$$

Approach

- compute harmonic function
- form conjugate pair

BUT…

Issues

Laplace needs boundaries…

- simple conjugacy ?
- natural boundaries
  - $E_C=E_D-A_R$
- Möbius symmetries!
- reflection boundary
  - double cover (Gu/Yau)

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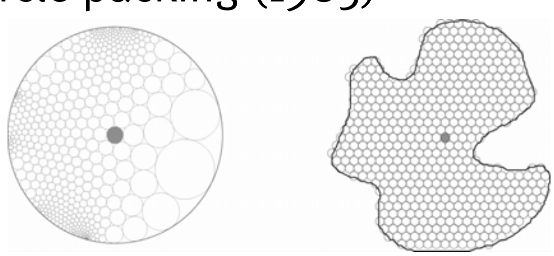
Circle Packings

Thurston's conjecture

- approximate Riemann map through circle packing (1985)



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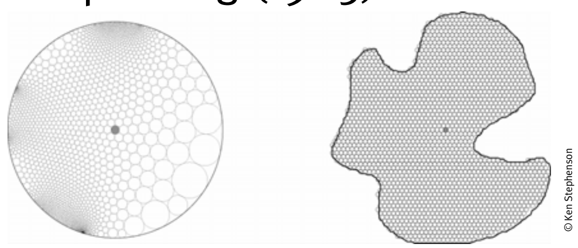
Circle Packings

Thurston's conjecture

- approximate Riemann map through circle packing (1985)



Circle Packings

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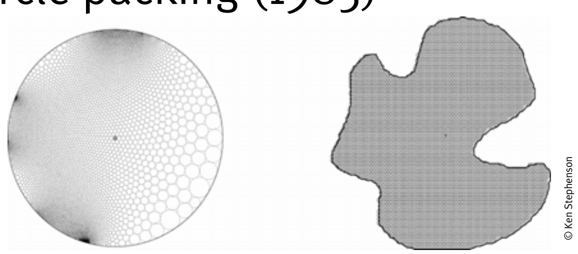


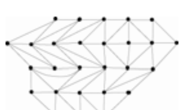
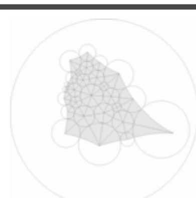
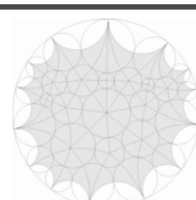
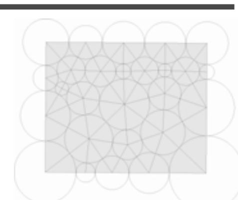
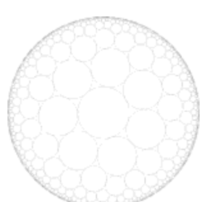
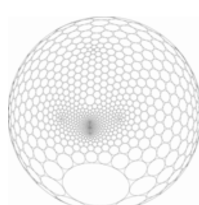
- proof: Rodin & Sullivan 1987

Circle Packing

Big picture idea

- continuous conformal maps:
  - infinitesimal circles to infinitesimal circles
- discrete version:
  - finite circles to finite circles
  - much of analytic function theory carries over (Stephenson)

Some Examples

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Algorithm

Given

- geometry: $\mathbb{C}$, $\mathbb{D}$, $\mathbb{S}^2$ (just a disk)
- combinatorics: 2-complex, (V,E,F)
  - with boundary for now
- boundary conditions

We are solving for radii, not positions!

Result

- packing label: radius r(v) for $v \in V$

Dirichlet Problem

Existence of solutions

Euclidean

Hyperbolic

- if K is a topological disk
- g: $\partial K \rightarrow (0,\infty)$ (or $(0,\infty]$) gives radii for the boundary vertices
- an angle target function
- $\Rightarrow \exists$ unique packing label which agrees with the boundary data

Coherent Angle Systems

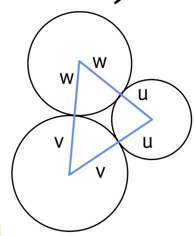
Radii give angles (and positions)

- Euclidean

$$\alpha(v;u,w) = 2\sin^{-1}\sqrt{\left(\frac{u}{v+u}\right)\left(\frac{w}{v+w}\right)}$$

- Hyperbolic

$$\alpha(v;u,w) = 2\sin^{-1}\sqrt{v\left(\frac{1-u}{1-vu}\right)\left(\frac{1-w}{1-vw}\right)}$$



$$\theta(v;R) = \sum_{(v,u,w)\in K} \alpha(v;u,w)$$ Angle sum

Angle target $A(v) = 2\pi n$

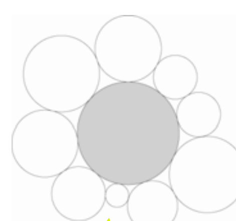
Angle Target Function

Notion of branch points

- combinatorial condition

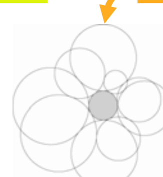
$$\sum_{v\in\gamma^{o}} (A(v) - 2\pi) \leq (E(\gamma) - 3)\pi$$

Closed chains of edges



n=1

n=2



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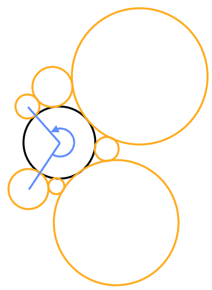
- packing condition

$$\theta(v;R) = A(v) = 2\pi n \qquad n \geq 1$$

Algorithm

Iterativelly adjust radii

- meet angle target: UNM method

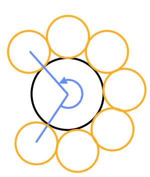


$$\theta(r_0; r_1, \ldots, r_k) = a \neq A(v)$$

Algorithm

Iteratively adjust radii

- meet angle target: UNM method



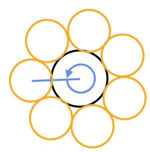
$$\theta(r_0; r_1, \ldots, r_k) = a \neq A(v)$$

$$\hat{\theta}(r_0; \hat{r}) = a$$

ALGORITHM

Iteratively adjust radii

- meet angle target: UNM method



$$\theta(r_0; r_1, \ldots, r_k) = a \neq A(v)$$

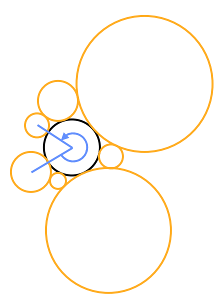
$$\hat{\theta}(r_0; \hat{r}) = a$$

$$\hat{\theta}(\tilde{r}_0; \hat{r}) = A(v)$$

ALGORITHM

Iteratively adjust radii

- meet angle target: UNM method



$$\theta(r_0; r_1, \ldots, r_k) = a \neq A(v)$$

$$\hat{\theta}(r_0; \hat{r}) = a$$

$$\hat{\theta}(\tilde{r}_0; \hat{r}) = A(v)$$

$$\theta(\tilde{r}_0; r_1, \ldots, r_k) = b \uparrow A(v)$$

Algorithm

Iteratively adjust radii

- meet angle target: UNM method
- simple direct calculation
- error goes down each time
- locally linear convergence
- simple acceleration possible
  - exploit strict monotonicity

Different Settings

Topology of K

- disk: hyperbolic packing
- sphere: punch hole, see above
- compact, $g=1$: Euclidean packing
  - scale not fixed
- compact, $g \geq 2$: Hyperbolic packing
  - unique

Generalizations

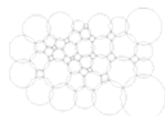
Boundary conditions

- angle targets and radii

Mixed boundary

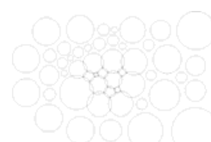
Tangency $\phi(i,j) \in [0,\pi/2]$

- tangency angles



Positive tangency angles

Imaginary tangency angles



Everything...

Back to History

Theory

- existence and uniquness on $\mathbb{S}^2$
  - Koebe 36; Andreev 70
- modern packings
  - Rudin & Sullivan 87 (hex packing)
  - He & Rodin 93 (finite valence)
  - He & Schramm 98 (C^∞ convgce.)

History

Algorithms

- Thurston 85
- Stephenson et al. 95, 03
- Mohar 93

Interesting connection with graph drawing and Tutte...

Variational approaches 91-02

- de Verdière, Brägger, Rivin, Leibon, Bobenko & Springborn

Main Results

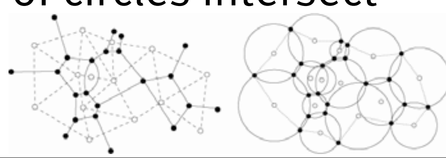
Koebe

- for every triangulation of the sphere there is a combinatorially equivalent circle packing; it is unique up to Möbius transformations
- generalization by Bobenko & Springborn: polytopal and higher-g

Main Results

Bobenko & Springborn

- for every polytopal decomposition of the sphere there is a unique (up to Möbius transformations) circle pattern; along primal/dual edge, pairs (quads) of circles intersect orthogonally.



Cellular Decomposition

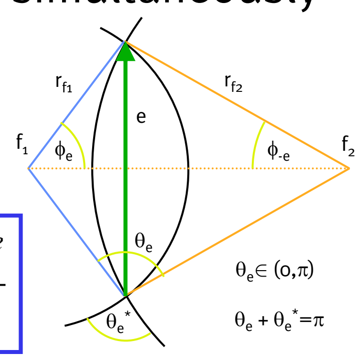
Quadgraphs

remember the "double"

- represent Σ and Σ^* simultaneously
- cone angles
  - vertex: $\Theta_v = \sum \theta_e$
  - face: $\Phi_f = 2\sum \phi_e$

Euclidean

$$\boxed{\phi_e = \frac{1}{2i}\log\frac{r_{f_j} - r_{f_k}e^{-i\theta_e}}{r_{f_j} - r_{f_k}e^{i\theta_e}}}$$



Existence & Uniqueness

Bobenko & Springborn (Thm. 3)

- Σ oriented cellular surface
- $\theta^* \in (0,\pi)^E$ and $\Phi \in (0,\infty)^F$
- unique packing exists $\Leftrightarrow$

(i) $$\sum_{f\in F} \Phi(f) = 2 \sum_{e\in E} \theta^*(e)$$

(ii) $$\sum_{f\in F'} \Phi(f) < 2 \sum_{e\in E'} \theta^*(e)$$

Hyperbolic

$\boxed{F' \subseteq F}$

$\boxed{F' \subset F}$

Euclidean

Proof by Construction

Variational formulation

- insight: $\rho = \begin{cases} \log r \\ \log \tanh(r/2) \end{cases}$

$$\boxed{\begin{aligned} S_E(\rho) = \ \sum_{f_j|e|f_k} \ & \left(-\theta_e^*(\rho_{f_j} + \rho_{f_k}) + \right. \\ & \operatorname{Im}\mathrm{Li}_2\left(e^{\rho_{f_k} - \rho_{f_j} + i\theta_e}\right) + \\ & \left. \operatorname{Im}\mathrm{Li}_2\left(e^{\rho_{f_j} - \rho_{f_k} + i\theta_e}\right)\right) \\ + \sum_f \ & \Theta_f \rho_f \end{aligned}}$$

Coherent Angle Systems

Definition: $\phi \in \mathbb{R}^E$

- $\forall$ faces: $\sum 2\phi_e = \Theta_f$
- $\forall$ oriented(!) edges:

$$\phi_e > 0, \quad \phi_e + \phi_{-e} = \theta_e^* \qquad \text{Euclidean}$$
$$\phi_e > 0, \quad \phi_e + \phi_{-e} < \theta_e^* \qquad \text{Hyperbolic}$$

- convex energies have critical point iff coherent angle system exists

Energy Minimum

$$f_\theta(x) = \frac{1}{2i} \log \frac{1 - e^{x - i\theta}}{1 - e^{x + i\theta}}$$

Critical point

- Euclidean case:

$$\sum_k 2 f_{\theta_{jk}}(\rho_k - \rho_j) = \Theta_j$$

- scaling ambiguity: fix $\sum \rho_j = 0$

Setup

- fix boundary face angles
- minimize (Hessian available)

Comparison

DEC approach

- simple computations!
- no real boundary control
- missing Möbius DOFs!

Circle packing

- where's the metric?
- boundary control better?

How can we get best of both worlds?