Towards Discrete Exterior Calculus

Mathieu Desbrun Anil Hirani Melvin Leok Jerry Marsden

USC/Caltech

Background

Interdisciplinary effort:

- Computer Science/Computer Graphics
- Applied Mathematics

Common denominator:

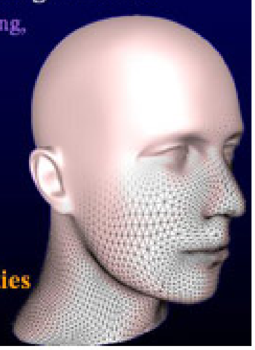
Applied Geometry

for reliable computations on discrete geometry

Motivation

Many applications require quantities from differential geometry, but on triangle meshes

- Smoothing, Simplification, Remeshing, Parameterization, Simulation,…
- Subdivision surfaces
- "Digital Geometry Processing"
  - Schrøder, Weldens, Zorin, Desbrun



Still no consensus on how to compute basic surface properties (normals, curvatures,…)

Overview

How to compute continuous quantities (e.g., curvatures) on a discrete surface?

- We could use polynomial reconstruction
  - Leads to oscillations (more so for irregular sampling)
  - Inconsistent view of the surface
- The mesh is often the only "reliable" data, therefore…

We proposed discrete operators satisfying discrete versions of continuous properties, using:

- Averaging Voronoi cells over the ***mesh itself***
- Mixed Finite Element / Finite Volume method
- [Meyer02]

Some Previous Work I

Global geometric quantities:

- Discrete Analogies of Continuous Quantities
  - Steiner polynomial, Minkowski's Quermassintegrale
    - Weyl, Hadwiger, Wintgen, Zähle, Fu, Morvan…
  - Integrals of geometric quantities: area, curvatures
- Harmonic map, Minimal surfaces…
  - Pinkall & Polthier
  - Defined as critical point of Dirichlet energy

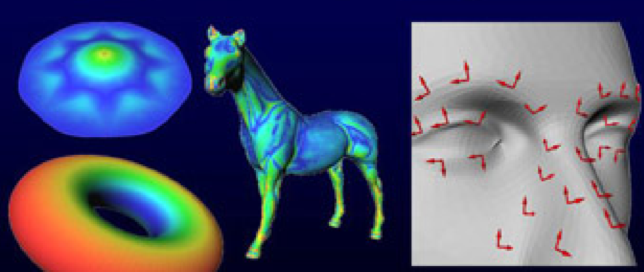
Some Previous Work II

Local geometric quantities:

- Normal = Weighted Average of Face Normals
  - Use incident angles as weights [Thurmer&Wurthrich '98]
  - Assume surface locally approximates a sphere [Max '99]
- Gaussian Curvature
  - Alekhsandrov, Polthier & Schmies, etc…
- Curvature Tensor Estimation
  - Weighted normal and tensor estimation [Taubin '95]
  - Least squares paraboloid fitting [Hamann '93, Clarenz *et al.* '00]

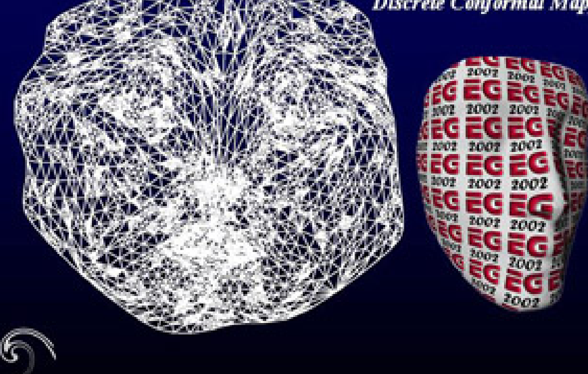
Few Relevant Results

Computing mean, Gaussian, and principal curvatures



Few Relevant Results

Discrete Conformal Mapping



Exterior Calculus

Foundation of calculus on smooth manifolds

- Historically, purpose was to extend div, curl, grad
- Basis of differential and integral computation
  - » Hodge decomposition, modern diff. geometry…
- A hierarchy of basic operators are defined:
  - » d, $*$, $\wedge$, $\flat$, $\#$, i_X, $\mathcal{L}_X$
- See [Abraham, Marsden, Ratiu] , ch. 6-7

Why "Discrete" Exterior Calculus

Foundation for discrete computations!

- basic discrete operators
  - » consistently derived
  - » easy to compute, given a discrete mesh
- Other computations easy using basic ops
  - » Ex: $\Delta = d\delta + \delta d$, $\nabla f = df^{\#}$, ...
- Hopefully, will provide the "natural" discrete computations we want:
  - » Convergence, consistent results, etc…

Previous Work

Driven by need for improved numerics

- Mimetic Differencing: [Shashkov,Hyman]
  - » Importance of adjoints, but mostly for 2D and quads
- Lots of work for EMF (Maxwell's equations)
  - » Discretization of fields and their equations
  - » Importance of "coherence" of derivation
- Few work on pure DEC:
  - » [Hiptmair; Dezin; Hydon]
  - » Chains/cochains for simplices/forms, notion of dual
  - » PWL interpolation of forms over simplices
- Discrete Mechanics [Lew, Marsden, Ortiz, West]

Let's get right to it

We will work on discrete manifolds

- » no notion of differential surface
- » 2D – "easy" case
- » 3D – very useful in graphics and simulation
- » any finite dimension

Basic assumptions:

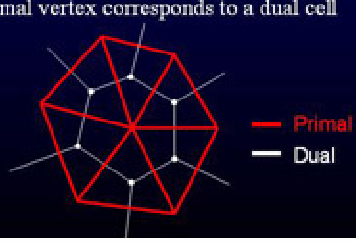
- arbitrary simplificial complex (called **primal mesh**)
- we will assume a "nice" triangulation for now
  - » no obtuse angle, for instance

Notion of Dual Mesh I

We use the circumcentric dual

In 2D:

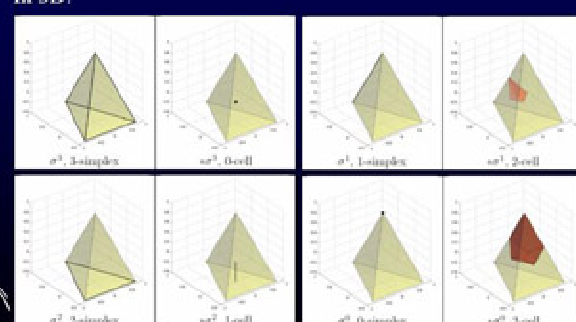
- A primal simplex corresponds a dual vertex
- A primal edge corresponds to a dual edge
- A primal vertex corresponds to a dual cell



Notion of Dual Mesh II

We use the circumcentric dual

In 3D:



Notion of Dual Mesh III

In general:

a primal σ^k corresponds to a dual $\hat{\sigma}^{n-k}$
(and vice-versa)

We will denote this mapping: ★

(Yes, it looks like a Hodge star)

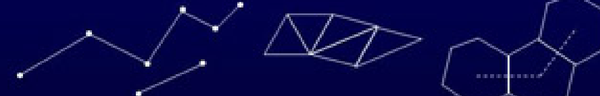
$$\star\sigma^k = \hat{\sigma}^{n-k}$$

$$\star\hat{\sigma}^k = \sigma^{n-k}$$

$$\star\star\sigma^k = (-1)^{k(n-k)}\,\sigma^k$$

Now, What is a Discrete Form?

- **Chain = set of cells (primal or dual)**



- **Form: cochain** (maps a number to a chain)
- **Pairing: integration**
  - Forms you heard about: dx, dy, dz, dA, dV etc….
  - a k-form needs a k-simplex to pair up with
  - $< \omega, \sigma > = \int_\sigma \omega$

Computing Forms on Meshes

Using linearity of integration:

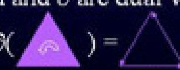
$$\int_{\bigcup_j \sigma_j} \omega = \sum_j \int_{\sigma_j} \omega$$

- split up k-chains into a set of simplicial k-chains
- a form is defined by its value on *each* simplex…
- no need for pointwise definition

Exterior Derivative

We deal with d by assuming Stokes' theorem:

$$\int_{\sigma} \mathrm{d}\omega = \int_{\partial\sigma} \omega$$

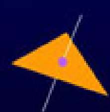
- simple "rewriting" rule
- d and ∂ are dual with respect to the natural pairing
- $\partial($  $)$
- careful orientations are needed, obviously

Hodge Operator

Important operator to "dualize" a form:

primal k-forms turn into dual (n-k)-forms

$$\frac{1}{|\star\sigma|}\int_{\star\sigma} \star\,\omega = \frac{1}{|\sigma|}\int_{\sigma}\omega$$



- well-known idea
- new evaluation, though…
- same *averages* – requires an induced metric
- forms can now be seen as living on central points

A Hands-on Derivation

$$\begin{aligned}
\Delta f(\mathbf{p}) &= -\int_{\sigma^0} \delta\, df \\
&= -\frac{|\sigma^0|}{|\ast\sigma^0|}\int_{\ast\sigma^0} d\ast df \\
&= -\frac{1}{|\ast\sigma^0|}\int_{\partial\ast\sigma^0} \ast df \\
&= -\frac{1}{|\ast\sigma^0|}\sum_i \frac{(\partial\ast\sigma^0)_i}{|\ast(\partial\ast\sigma^0)_i|}\int_{\ast(\partial\ast\sigma^0)_i} df \\
&= -\frac{1}{|\ast\sigma^0|}\sum_i \frac{|VE_i|}{|E_i|}\left(f(\mathbf{p}) - f(\mathbf{q}_i)\right) \\
&= -\frac{1}{|\ast\sigma^0|}\sum_i \left(\cot\alpha_i + \cot\beta_i\right)\left(f(\mathbf{p}) - f(\mathbf{q}_i)\right)
\end{aligned}$$

Laplace-Beltrami as defined in [DMSB99]

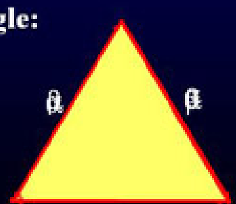
using variational approach…

Wedge Product I

Purely combinatorial definition of $\wedge$:

$$\int_{\sigma^{(k+l)}} \alpha^k \wedge \beta^l = \frac{1}{(k+l)!}\sum_{\tau\in S^{k+l+1}} \mathrm{sign}(\tau)\frac{|\sigma^{k+l}\cap \ast v_{\tau(k)}|}{|\sigma^{k+l}|}\int_{[v_{\tau(0)},\ldots,v_{\tau(k)}]}\alpha \int_{[v_{\tau(k)},\ldots,v_{\tau(k+l)}]}\beta$$

On a triangle:



Wedge Product II

Purely combinatorial definition of $\wedge$:

$$\int_{\sigma^{(k+l)}} \alpha^k \wedge \beta^l = \frac{1}{(k+l)!}\sum_{\tau\in S^{k+l+1}} \mathrm{sign}(\tau)\frac{|\sigma^{k+l}\cap \ast v_{\tau(k)}|}{|\sigma^{k+l}|}\int_{[v_{\tau(0)},\ldots,v_{\tau(k)}]}\alpha \int_{[v_{\tau(k)},\ldots,v_{\tau(k+l)}]}\beta$$

- Comes in two flavors:
  - Primal x Primal ➔ Primal
  - Dual x Dual ➔ Dual
- Try it on a triangle for dx $\wedge$ dy if you want!
- Reminiscent of Crofton's formula…

Wedge Product III

Properties respected:

- Anti-commutativity (simple)
- Associativity (unsure, maybe not true ☺)
- Leibniz rule (fairly simple)

- No need for interpolated forms
  - no barycentric coordinates involved
- Could be used as the "real" definition of the Hodge star

Vector Fields

We need to map vectors and 1-forms, and vice-versa

- *b*: vector field into a 1-form
- #: 1-form into a vector field

Natural definition of a vector field:

Average vector values over cells

- lives at dual 0-simplices (primal vector fields)
- or at primal 0-simplices (dual vector fields)

Vector Fields

Divergence and Curl

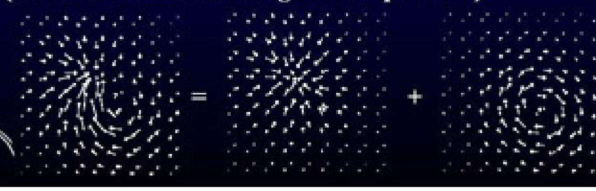
Once b and # is defined, we have:

$$div\ \mathbf{V} = \star d \star \mathbf{V}^{b}$$

$$curl\ \mathbf{V} = (\star d\mathbf{V}^{b})^{\#} \qquad \text{(in 3D...)}$$

Interesting connection:

Straightforward variational approach [Polthier]
(discrete Helmhotz-Hodge decomposition)



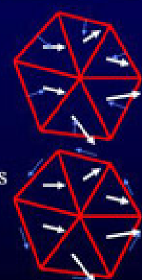
Interpretation of Div and Curl

Divergence:

- "Flux" through the Voronoi cell

Curl:

- "Circulation" on Voronoi boundaries



Implications:

- b: primal 0 → dual 1, *or* dual 0 → primal 1
- #: dual 1 → primal 0, or primal 1 → dual 0

Contraction of Form

Home-brewed formula:

$$i_{\mathbf{X}}\omega = (-1)^{k(n-k)} \star (\star\omega \wedge \mathbf{X}^{b})$$

- dualize the form, add a "slot" by wedging it, then dualize it back: a slot is taken…
- no need for extra operator
- only primal/dual or dual/primal

Lie Derivative

Just use Cartan magic formula:

$$\mathcal{L}_{\mathbf{X}}\omega = i_{\mathbf{X}} d\omega + d i_{\mathbf{X}}\omega$$

- still not tested
- but should satisfy all continuous properties
- should also lead to advection on discrete meshes
- flows are next…

Recap

Discrete Exterior Calculus

- Averaged integrals – seems to be *the* key
- Purely combinatorial, no interpolation needed
- Subtle interplay between primal and dual
- Simple "rewriting" rules → mathematica code?
- Seems like all basic ops carry over just fine

- Almost complete…

A Word About Numerics

Mesh quality:

- » clearly, Delaunay criterion makes numerics better
  - Otherwise, there are some negative weightings
  - But global integrals remain fine!
  - So only local degeneracy happens
- » linear accuracy if mesh is of poor quality
- » super-accuracy if symmetries

Full-blown accuracy study needed

- » convergence study

What's next

- Differential geometry:
  - » Defining **n**, shape operator, etc…
- Multigrid methods
  - » Remeshing cochains, mapping between manifolds
- Discrete Connections
- Piecewise linear shells (w/ Peter and Eitan)
- Revisiting old techniques (primal/dual)?
  - » Staggered grids in fluid simulation
  - » Leapfrog integration
  - » …
- Develop a discrete 'know-how'…