Low-dimensional models of plane Couette flow using the proper orthogonal decomposition

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with Jeff Moehlis and Philip Holmes

Someone remarked to me once: ”Physicians shouldn’t say, I have cured this man, but, this man didn’t die under my care.” In physics, too, instead of saying, I have explained such and such a phenomenon, one might say, I have determined causes for it the absurdity of which cannot be conclusively proved. Georg Christoph Lichtenberg, 1742-1799
Talk Outline

- **Plane Couette flow (PCF)**
  - Flow geometry and equations
  - Properties of PCF

- **Low-dimensional modelling**
  - Historical examples
  - The proper orthogonal decomposition
  - Galerkin projection

- **The minimal flow unit**
  - Model 1: 6 mode coupled (11-d)
  - Model 2: 9 mode uncoupled (16-d)

- **Comparisons with the models of Waleffe**

- **Conclusion**
plane Couette flow (PCF)

**Flow geometry and equations**

Nondimensional equations for perturbation $[\mathbf{u}, p] = [(u, v, w)^T, p]$ to laminar state:

$$\frac{\partial}{\partial t} \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - y \frac{\partial}{\partial x} \mathbf{u} - u_2 \mathbf{e}_x - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u},$$

$$\nabla \cdot \mathbf{u} = 0, \quad u|_{y=\pm d/2} = 0, \quad Re = U_0 d/(2\nu)$$

Periodic Boundary Conditions (PBCs) in $x$ and $z$
Properties of PCF

- laminar state linearly stable for all $Re$
- turbulence at high $Re$ and/or perturbation amplitudes
- unstable, steady finite amplitude solutions
  - consist of streamwise vortices and streaks
  - do not bifurcate from the laminar state

Some of the big questions

- how does this “sub-critical transition” to turbulence come about?
- importance of unstable, steady solutions?
- can the turbulent state be described with a simple model?
**Low-dimensional modelling**

***Historical Overview - Theoretical***

- Naver-Stokes PDE is *infinite*-dimensional
- Mid-20th century: Hopf imagined a finite-dimensional attractor - one which came about from successive bifurcations to periodic and quasiperiodic attractors
- 1970’s: Ruelle & Takens $\rightarrow$ *strange attractor*
- 1980’s: Constantin, Foias, Temam found finite bounds on dimensions of attractors for 2-d Navier-Stokes and the 1-d Kuramoto-Sivashinksy Equation
Low-dimensional modelling

Historical Overview - Numerical

- **1963:** Lorenz derived his famous ODEs through modelling Rayleigh-Bénard convection
  - ODEs produced through projection of PDE onto subspace defined by sinusoidal basis functions
  - Mathematically enchanting but a poor physical model

- **1968:** Lumley proposed Proper Orthogonal Decomposition as a technique for flow description
  - Instead of sinusoidal modes find basis optimal for particular flow

- **1987:** Aubry, Holmes and others
  - ODEs produced through projection of PDE onto subspace defined by POD modes
  - Convincing qualitative model, later questioned by Gibson (2002)
Low-dimensional modelling

The proper orthogonal decomposition
(c.f. Karhunen-Loeve decomposition, PCA, SVD)

- consider set of velocity snapshots \( \{ u(x) \} \)
  \( \langle \cdot \rangle \) = averaging operation over snapshots
  \( (\cdot, \cdot) = L^2 \) inner product (norm: \( \| \cdot \| \))
- want to find basis of POD modes \( \{ \Phi \} \) which maximizes
  \[
  \frac{\langle \| (u, \Phi) \|^2 \rangle}{\| \Phi \|^2}
  \]
  i.e., find extremes of \( J[\Phi] = \langle \| (u, \Phi) \|^2 \rangle - \lambda (\| \Phi \|^2 - 1) \)

\[
\rightarrow \sum_{j=1}^{3} \int \int \int_{\Omega} \langle u_i(x, t)u_j^*(x', t) \rangle \Phi_{j_{nx}nxz}^{(n)}(x')d^3x' = \lambda_{nxnxz}^{(n)} \Phi_{inxnxz}^{(n)}(x)
\]
  \( i = 1, 2, 3 \)
**Properties of POD Modes**

- orthogonality (normalization $\rightarrow$ orthonormality)
- (almost) every member of $\{u(x)\}$ is a linear combination of POD modes
- POD modes individually satisfy incompressibility, BCs
- optimality: for a given number of modes energy captured by POD basis $> \text{any other basis}$
  - eigenvalue $\lambda$ represents twice average KE in POD mode
  - subspace spanned by the modes corresponding to the largest eigenvalues captures the most energetic disturbances
Low-dimensional modelling

**POD applied to PCF - Symmetry Considerations**

- Fourier modes optimal for directions with translation symmetry
  \[
  \Phi_{n_xn_z}^{(n)}(x) = \frac{\phi_{n_xn_z}^{(n)}(y)}{\sqrt{L_xL_z}} \exp \left( 2\pi i \left( \frac{n_xx}{L_x} + \frac{n_zz}{L_z} \right) \right)
  \]

- enlarge set of snapshots with discrete symmetries
  \[
  \mathcal{P} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] = (-u_1, -u_2, -u_3, p)(-x, -y, -z, t)
  \]
  \[
  \mathcal{R} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] = (u_1, -u_2, u_3, p)(x, -y, z, t)
  \]
  \[
  \mathcal{RP} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] = (-u_1, u_2, -u_3, p)(-x, y, -z, t)
  \]

- P is a point rotation about the centre of the system
- R is a reflection in the central streamwise-wallnormal \((x - y)\) plane \((z = 0)\)
Galerkin Projection

**Galerkin projection**

- write evolution PDE as
  \[ \frac{\partial u}{\partial t} = F(u) \]

- expand
  \[ u(x, t) = \sum_{n=1}^{\infty} \sum_{n_x=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} a_{n_x,n_z}(t) \Phi_{n_x,n_z}(x), \quad a_{n_x,n_z} \in \mathbb{C} \]

- substitute into evolution PDE
  \[ \sum_{n,n_x,n_z} \dot{a}_{n_x,n_z}(t) \Phi_{n_x,n_z}(x) = F \left( \sum_{n,n_x,n_z} a_{n_x,n_z}(t) \Phi_{n_x,n_z}(x) \right) \]

- take inner product with \( \Phi_{m_x,m_z}(x) \), use orthonormality
  \[ \Rightarrow \dot{a}_{m_x,m_z}(t) = \left( F \left( \sum_{n,n_x,n_z} a_{n_x,n_z}(t) \Phi_{n_x,n_z}(x) \right), \Phi_{m_x,m_z}(x) \right) \]
Minimal flow unit: definition

... the smallest domain in which turbulence can be sustained

Important references

- **channel flow: Jiminez and Moin (JFM, 1991)**
  - found spanwise width \( \approx 100 \) wall units required to sustain turb.
  - cf. Kline et al, JFM 1967; \( 100 \approx \) streak spacing in turb. boundary layer

- **PCF: Hamilton et al (JFM, 1995) - “HKW”**
  - found analogous minimal flow unit for PCF
  - at \( Re = 400 \) domain size was \( L_x \times L_z = 1.75\pi \times 1.2\pi \)
  - suggested periodic orbit \( \approx \) “backbone” for turbulence
  - inspired several low-d models (Waleffe, Phys. Fluids 1995/7)

- **PCF: Kawahara & Kida (JFM, 2001) - ”K & K”**
  - examined minimal flow unit as defined by HKW
  - discovered an *unstable* periodic orbit buried inside DNS
Minimal flow unit: simulation

**Numerical database**
- basic channel code provided by C. Rowley
- algorithm: Kim, Moin & Moser, JFM 1987
  - pseudospectral spatial discretisation
  - Crank-Nicholson routine for linear terms
  - 3rd Order Runge-Kutta for nonlinear terms
- grid: $N_x = N_z = 16$ Fourier moves
  - 33 Chebyshev Polynomials in $y$ direction (cf. HKW, K & K)
- sustained turb. at $Re = 400$ difficult to achieve
  - begin with random state at $Re = 625$
  - ran until converged, reduced $Re$ and repeated
- integrated time $= 20,000$ outer time units
- one sample saved every 5 time units (4000 total)
Minimal flow unit: simulation

**Turbulent statistics comparisons**

Mean streamwise velocity

RMS turbulent fluctuations

\[\langle u \rangle \quad \text{[KK01]}\]
Minimal flow unit: phenomenology

For PCF in the MFU ∃ a “self-sustaining process”

- define modal RMS velocity as

\[
M(n_x, n_z) \overset{\text{def}}{=} \left( \int_{-1}^{1} \left[ \tilde{u}_1^2(n_x, y, n_z) + \tilde{u}_2^2(n_x, y, n_z) + \tilde{u}_3^2(n_x, y, n_z) \right] dy \right)^{1/2}
\]

Note flow is roughly periodic - with period \(80 \approx 100\) (cf. HKW Fig 3-a)
Minimal flow unit: phenomenology

The process of streak breakdown

Midplane $x$ velocity contours - top view

(cf. HKW Fig. 2)
Minimal flow unit: phenomenology

The $x$-independent modes & streak breakdown

Before Breakdown

After Breakdown

N.B. plots on the left include the laminar profile (cf. HKW Fig. 4)
Minimal flow unit: objectives

- PCF-MFU: a test-bed for low-d models
  - "low-order" tests
    - approximate reproduction of energy budgets
    - evolution of model amplitudes $\sim$ evolution of DNS amplitudes
    - approximate reproduction of time-scales
  - "high-order" tests
    - model velocity field reconstructions similar to DNS
    - maintain important properties of flow being considered
      eg. for PCF, stability of the laminar state
    - admits a plausible bifurcation scenario - transition to turbulence
## Minimal flow unit: analysis

### POD decomposition of PCF-MFU

<table>
<thead>
<tr>
<th>$(n, n_x, n_z)$</th>
<th>$\lambda_{n_x,n_z}^{(n)}$</th>
<th>$%E_{n_x,n_z}^{(n)}$</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 0, 0)$</td>
<td>4.4550</td>
<td>68.02</td>
<td>□</td>
</tr>
<tr>
<td>$(1, 0, \pm 1)$</td>
<td>0.7821</td>
<td>23.88</td>
<td>□</td>
</tr>
<tr>
<td>$(1, 0, \pm 2)$</td>
<td>0.0543</td>
<td>1.66</td>
<td>□</td>
</tr>
<tr>
<td>$(1, \pm 1, 0)$</td>
<td>0.0386</td>
<td>1.18</td>
<td>□</td>
</tr>
<tr>
<td>$(1, 0, \pm 3)$</td>
<td>0.0195</td>
<td>0.59</td>
<td></td>
</tr>
<tr>
<td>$(2, 0, 0)$</td>
<td>0.0174</td>
<td>0.27</td>
<td>“exclude $n=2$”</td>
</tr>
<tr>
<td>$(2, 0, \pm 1)$</td>
<td>0.0123</td>
<td>0.38</td>
<td>“exclude $n=2$”</td>
</tr>
<tr>
<td>$(1, \pm 1, \pm 2)$</td>
<td>0.0109</td>
<td>0.33</td>
<td></td>
</tr>
<tr>
<td>$(1, \pm 1, \pm 1)$</td>
<td>0.0090</td>
<td>0.27</td>
<td>□</td>
</tr>
</tbody>
</table>

...  

- choose $(1, \pm 1, \pm 1)$ instead of $(1, \pm 1, \pm 2)$ since the former appears to be more important in the streak formation/breakdown process
Minimal flow unit: analysis

Some of the POD modes

(1, 0, 1) mode

(1, 0, 2) mode

(1, 0, 3) mode

(1, 1, 0) mode
Constructing a model from the modes 
\((n, n_x, n_z) = (1, 0, 0), (1, 0, 1), (1, 0, 2), (1, 1, 0), (1, 1, \pm 1)\)
→ a set of 6 ODEs (1 real + 5 complex) of the form

\[
\begin{align*}
\dot{a}_{0,0}^{(1)} &= -A_{0,0}^{(1)} a_{0,0}^{(1)} + \sum B_{n_x,n_z}^{(1)} |a_{n_x,n_z}^{(1)}|^2 \\
\dot{a}_{n_x,n_z}^{(1)} &= (A_{n_x,n_z}^{(1)} - B_{n_x,n_z}^{(1)} a_{0,0}^{(1)}) a_{n_x,n_z}^{(1)} + N_{n_x,n_z}^{(1)} \\
N_{n_x,n_z}^{(1)} &\equiv C_{n_x,n_zm_xm_z} a_{m_x,m_z}^{(1)} a_{n_x-m_x,n_z-m_z}^{(1)}
\end{align*}
\]

- only keep one “quantum” number per Fourier pair  
  \(\implies\) no interaction at the linear level

- The 3 ODEs corresponding to interaction between 
  \((n, n_x, n_z) = (1, 0, 0), (1, 0, 1), (1, 0, 2)\) modes comprises the 0:1:2 resonance (article in preparation)

- seek to compare the ODE models to the DNS; 
  projection of DNS onto \((n, n_x, n_z)\) denoted by \(\hat{a}_{n_x,n_z}^{(n)}\)
Minimal flow unit: coupled models

- Projection of DNS onto 3 most energetic POD modes

- $(1, 0, 1), (1, 0, 2)$ move around a circle in a slow, chaotic fashion - relatively fast motion in the radial direction
- $(1, 1, 0), (1, 1, \pm 1)$ more complex, of less energetic importance
- model caricatures this complex behaviour with a simple travelling wave
Minimal flow unit: coupled models

How many modes would be required for accurate reproduction of DNS?
- \( \approx 1000 \), as these 600 mode projections indicate...
Effect of neglected terms

- Terms neglected through truncation which are small on average have a significant effect.
- Exact equations for the evolution of the modal amplitudes are of form

\[ \dot{a}_{n_x n_z}(t) = \text{set of retained terms} + T_{n_x n_z}^{(n)} \]

- Simplest possible approximation is of the form

\[ T_{n_x n_z}^{(n)} = -\beta_{n_x n_z}^{(n)} a_{n_x n_z}^{(n)} \]

(Heisenberg Eddy-viscosity model)

- Choose \( \beta_{n_x, n_z}^{(n)} \) through least-squares fit.
How well does this approximation work?

- For some modes, it works very well...
  - eg. the (1,0,2) mode
Minimal flow unit: coupled models

- For others it does not work very well at all...
  - eg. the (1,0,1) mode

\[
\begin{align*}
\text{dotted: } & \Re(T_{0,1}^{(1)}) \\
\text{solid: } & \Re(-10\beta_{0,1}^{(1)}\hat{a}_{0,1}^{(1)}) \\
\text{dotted: } & \Im(T_{0,1}^{(1)}) \\
\text{solid: } & \Im(-10\beta_{0,1}^{(1)}\hat{a}_{0,1}^{(1)})
\end{align*}
\]
Minimal flow unit: coupled models

- attempt to average out this variability with a spectral transfer model

\[ \dot{a}_{n_x,n_z}^{(n)} = \cdots - \alpha \nu (n_x^2 + n_z^2) a_{n_x,n_z}^{(n)} \]

with

- \( \nu = 0.0333 \) (computed from weighted average of \( \beta \)s)
- \( \alpha = \) an \( \mathcal{O}(1) \) bifurcation parameter

- Note: no term added to the equation for \( \dot{a}_{0,0}^{(1)} \)

- fix \( Re = 400 \), examine behavior as \( \alpha \) varies
Minimal flow unit: coupled models

- **Bifurcation diagram**

\[ \alpha = 0 \quad \alpha = 1 \]

- Stable
- Unstable

fp = fixed point

TW

SW1

MW

SW2
Minimal flow unit: coupled models

- DNS vs. model at $Re = 400$, $\alpha = 0.8$

- Instead of travelling waves now have *standing* waves

- Amplitude of $a_{0,0}^{(1)}$ far too low, others aren’t too bad
Minimal flow unit: coupled models

- **DNS vs. model RMS modal velocities**

Representative cycle from the DNS

Coupled 6-mode model ($\alpha = 0.8$)

Period $\approx 80 - 100$ outer time units

Period $\approx 95$ outer time units
Minimal flow unit: coupled models

- DNS vs. model streak-breakdown process

Reconstructed from DNS

Coupled 6-mode model ($\alpha = 0.8$)
Minimal flow unit: coupled models

- DNS vs. model streamwise velocity contours (including laminar state)

Reconstructed from DNS

Coupled 6-mode model ($\alpha = 0.8$)

Before Breakdown

After Breakdown
Minimal flow unit: coupled models

- DNS vs. model streamwise velocity contours (neglecting laminar state)

Reconstructed from DNS

Coupled 6-mode model ($\alpha = 0.8$)

Before Breakdown

After Breakdown
Minimal flow unit: coupled models

DNS vs. 6-mode model turbulent statistics

\[ \sqrt{\langle u'^2 \rangle} \]

\[ \langle u' v' \rangle \]
Minimal flow unit: uncoupled models

- Model captures MFU turbulence as periodic orbit
  - approximately correct time-scales, phase relationships, and velocity reconstructions

- But, predicts laminar state becomes unstable
  \[ \Rightarrow \text{uncoupled expansion} \]

\[
\Phi^{(n)}_{n_xn_z}(x) = \Phi^{(n)[1]}_{n_xn_z}(x) + \Phi^{(n)[2]}_{n_xn_z}(x)
\]

\[
\mathbf{u}(x, t) = \sum_{n=1}^{\infty} \sum_{n_x=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \sum_{m=1}^{2} b^{(n)[m]}_{n_xn_z}(t) \Phi^{(n)[m]}_{n_xn_z}
\]

- first suggested to avoid paradoxical behaviour of ODE models based upon *streamwise-invariant* POD modes
Minimal flow unit: uncoupled models

Streamwise-invariant $\implies$ perturbations must decay!

- for mean velocity $U$, perturbations $u, v, w$ and $\partial/\partial x = 0$ we have

\[
D/Dt = \partial/\partial t + u_2 \partial/\partial y + u_3 \partial/\partial z \quad \text{(convective derivative)}
\]
\[
\frac{D}{Dt}(U + u_1) = \frac{1}{Re} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (U + u_1) \quad \text{(NS Equation)}
\]
\[
\frac{d}{dt} \int \int (u_2^2 + u_3^2) \, dydz = -2\nu \left\{ \int \int \omega_x^2 \, dydz \right\} \quad (\omega_x = x\text{-vorticity})
\]

$\Rightarrow$ all disturbances must decay

- “artificial coupling” $\Rightarrow$ streamwise-invariant model sustained dynamics
- Berkooz, Holmes & Lumley (JFM 1991) proposed to remove coupling
- later generalised by Waleffe (Phys. Fluids 1997)
Minimal flow unit: uncoupled models

Uncoupled basis functions

- Streamwise-invariant:
  \[
  \phi_{0,nz}^{(n)[1]}(y) = \begin{pmatrix}
  \phi_{1,0,nz}(y) \\
  0 \\
  0
  \end{pmatrix},
  \phi_{0,nz}^{(n)[2]}(y) = \begin{pmatrix}
  0 \\
  \phi_{2,0,nz}(y) \\
  \phi_{3,0,nz}(y)
  \end{pmatrix},
  \]

- Spanwise-invariant:
  \[
  \phi_{n_x,0}^{(n)[1]}(y) = \begin{pmatrix}
  \phi_{1,n_x,0}(y) \\
  \phi_{n}^{(n)}(y) \\
  \phi_{2,n_x,0}(y)
  \end{pmatrix},
  \phi_{n_x,0}^{(n)[2]}(y) = \begin{pmatrix}
  0 \\
  0 \\
  \phi_{3,n_x,0}(y)
  \end{pmatrix},
  \]

- New basis functions: pairwise orthogonal & divergence free
  \[
  (\Phi_{n_x,nz}^{(n)[m]}(x), \Phi_{n_x,nz}^{(n')[m']}(x)) = e_{n_x,nz}^{(n)[m]} \delta_{nn'} \delta_{mm'}
  \]
  \[
  \nabla \cdot \Phi_{n_x,nz}^{(n)[m]}(x) = 0 \text{ for } m = 1, 2
  \]

- Do not normalise uncoupled modes \( e_{n_x,nz}^{(n)[m]} \neq 1 \)
  \[
  \text{however: } e_{n_x,nz}^{(n)[1]} + e_{n_x,nz}^{(n)[2]} = 1
  \]
Minimal flow unit: uncoupled models

- Uncoupled 6-mode model

  - Model is not streamwise invariant
    $\implies$ might expect sustained behaviour

Plot RMS modal velocities: $M(n_x, n_z) = \frac{1}{\sqrt{L_x L_z}} \sum_n |a_{n_x, n_z}^{(n)}|$
Uncoupled 9-mode model

- Laminar solution is now stable with $\alpha > 0.2179$
  - small $\|a^{UC}(0)\|$ → laminar solution as $t \to \infty$
  - larger $\|a^{UC}(0)\|$ → periodic orbit, originating in an SN bifurcation

AUTO → periodic orbit originates in a SN bifurcation, forming an isola
Minimal flow unit: uncoupled models

□ DNS vs. model RMS modal velocities

Representative cycle from the DNS

Uncoupled 9-mode model ($\alpha = 0.22$)

Period $\approx 80 - 100$ outer time units

Period $\approx 92$ outer time units
Minimal flow unit: uncoupled models

- DNS vs. model streak-breakdown process

Reconstructed from DNS

Uncoupled 9-mode model ($\alpha = 0.22$)
Minimal flow unit: uncoupled models

- DNS vs. model streamwise velocity contours (including laminar state)

Reconstructed from DNS

Uncoupled 9-mode model ($\alpha = 0.22$)

Before Breakdown

After Breakdown
Minimal flow unit: uncoupled models

- DNS vs. 9-mode model turbulent statistics
Minimal flow unit: uncoupled models

Branches of fixed points

From Schmiegel (PhD thesis)

AUTO analysis of uncoupled 9-mode model

- was inspired by HKW to construct a low-dimensional model from physical arguments

- not PCF but “sinusoidal shear flow”

- predictions of 4-mode model
  - correctly predicts stability of laminar state
  - certain coefficients: stable p.o. through *homoclinic* bifurcation.
  - velocity reconstructions from p.o. do not look like DNS
  - coefficients from Galerkin projection of NSE → *unstable* p.o.
Conclusions

- **Have considered PCF in the MFU**
  - applied the POD - found almost all energy captured by first few modes
  - despite this, found that $\approx 1000$ modes required to reproduce DNS

- **Constructed two low-dimensional models**
  - 6-mode coupled model; qualitatively correct periodic orbit, but laminar state was *unstable*
  - 9-mode uncoupled model; periodic orbit co-existing with stable laminar state
  - both models showed good quantitative agreement with DNS time-scales, RMS modal velocities
  - good qualitative agreement with DNS streak-breakdown contours and statistics
  - Phase relationships of modal amplitudes better for coupled 6-mode model
Possible improvements

- **including more modes**
  - e.g., more POD families, linear terms → non-normal
  - can also solve the problem of the unstable laminar state

- **using symmetry ideas to get “better” POD modes**
  - Rowley and Marsden (Physica D, 2000)

- **better model for losses to neglected modes**