# Low-dimensional models of plane Couette flow

## using the proper orthogonal decomposition

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*with* ***Jeff Moehlis and Philip Holmes***

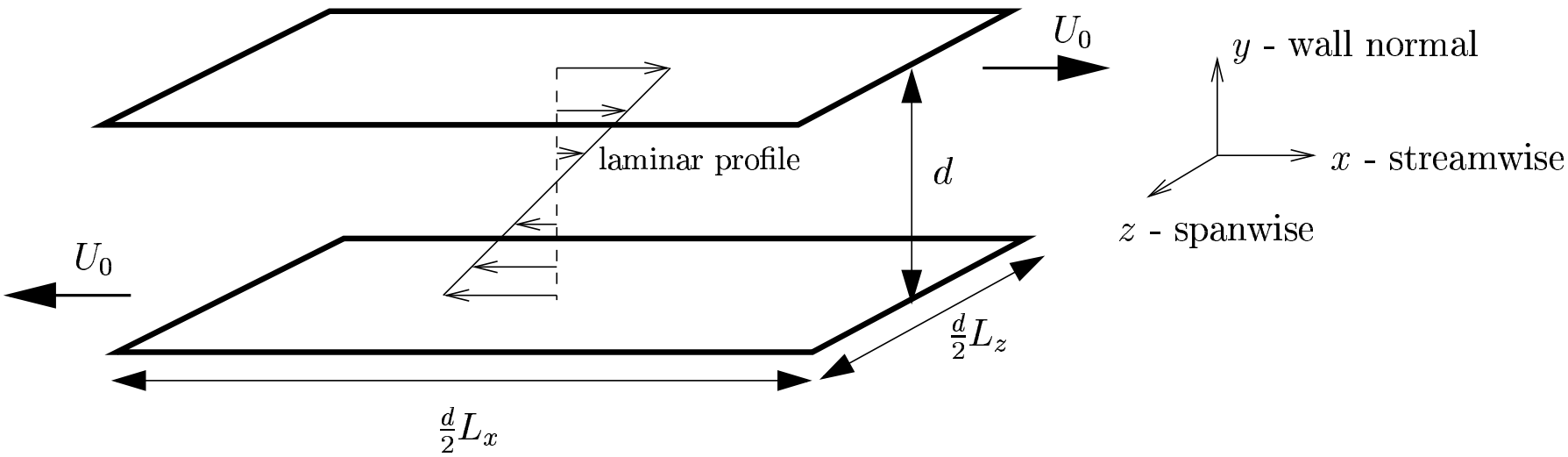
**Someone remarked to me once: "Physicians shouldn't say, I have cured this man, but, this man didn't die under my care." In physics, too, instead of saying, I have explained such and such a phenomenon, one might say, I have determined causes for it the absurdity of which cannot be conclusively proved.** *Georg Christoph Lictenberg, 1742-1799*

# Talk Outline

- **Plane Couette flow (PCF)**
  - Flow geometry and equations
  - Properties of PCF
- **Low-dimensional modelling**
  - Historical examples
  - The proper orthogonal decomposition
  - Galerkin projection
- **The minimal flow unit**
  - Model 1: 6 mode coupled (11-d)
  - Model 2: 9 mode uncoupled (16-d)
- **Comparisons with the models of Waleffe**
- **Conclusion**

# plane Couette flow (PCF)

## *Flow geometry and equations*




Nondimensional equations for perturbation $[\mathbf{u}, p] = [(u, v, w)^T, p]$ to laminar state:

$$\frac{\partial}{\partial t}\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - y\frac{\partial}{\partial x}\mathbf{u} - u_2\mathbf{e}_x - \nabla p + \frac{1}{Re}\nabla^2\mathbf{u},$$
$$\nabla\cdot\mathbf{u} = 0, \qquad u|_{y=\pm d/2} = 0, \qquad Re = U_0 d/(2\nu)$$

Periodic Boundary Conditions (PBCs) in $x$ and $z$

# plane Couette flow (PCF)

- Properties of PCF
  - **laminar state linearly stable for all $Re$**
  - **turbulence at high $Re$ and/or perturbation amplitudes**
  - **unstable, steady finite amplitude solutions**
    - consist of streamwise vortices and streaks
    - do not bifurcate from the laminar state
- Some of the big questions
  - **how does this "sub-critical transition" to turbulence come about?**
  - **importance of unstable, steady solutions?**
  - **can the turbulent state be described with a simple model?**

# Low-dimensional modelling

## *Historical Overview - Theoretical*

- Naver-Stokes PDE is *infinite*-dimensional
- Mid-20th century: Hopf imagined a finite-dimensional attractor - one which came about from sucessive bifurcations to periodic and quasiperiodic attractors
- 1970's: Ruelle & Takens $\rightarrow$ *strange attractor*
- 1980's: Constantin, Foias, Temam found finite bounds on dimensions of attractors for 2-d Navier-Stokes and the 1-d Kuramoto-Sivashinksy Equation

# Low-dimensional modelling

## *Historical Overview - Numerical*

- **1963: Lorenz derived his famous ODEs through modelling Rayleigh-Bénard convection**
  - ODEs produced through projection of PDE onto subspace defined by *sinusoidal* basis functions
  - mathematically enchanting but a poor physical model
- **1968: Lumley proposed Proper Orthogonal Decomposition as a technique for flow description**
  - instead of sinusoidal modes find basis optimal for particular flow
- **1987: Aubry, Holmes and others**
  - ODEs produced through projection of PDE onto subspace defined by POD modes
  - convincing *qualitative* model, later questioned by Gibson (2002)

# Low-dimensional modelling

## *The proper orthogonal decomposition*

*(c.f. Karhunen-Loeve decomposition, PCA, SVD)*

- consider set of velocity snapshots $\{\mathbf{u}(\mathbf{x})\}$
  $\langle\cdot\rangle$ = averaging operation over snapshots
  $(\cdot,\cdot) = L^2$ inner product (norm: $\|\cdot\|$)
- want to find basis of POD modes $\{\mathbf{\Phi}\}$ which maximizes

$$\frac{\langle|(\mathbf{u},\mathbf{\Phi})|^2\rangle}{\|\mathbf{\Phi}\|^2}$$

- i.e., find extremes of $J[\mathbf{\Phi}] = \langle|(\mathbf{u},\mathbf{\Phi})|^2\rangle - \lambda(\|\mathbf{\Phi}\|^2 - 1)$

$$\rightarrow \sum_{j=1}^{3} \int\int\int_{\Omega} \langle u_i(\mathbf{x},t) u_j^*(\mathbf{x}',t)\rangle \Phi^{(n)}_{jn_xn_z}(\mathbf{x}') d^3\mathbf{x}' = \lambda^{(n)}_{n_xn_z} \Phi^{(n)}_{in_xn_z}(\mathbf{x})$$

$$i = 1,2,3$$

# Low-dimensional modelling

## *Properties of POD Modes*

- **orthogonality (normalization $\rightarrow$ orthonormality)**
- **(almost) every member of $\{\mathbf{u}(\mathbf{x})\}$ is a linear combination of POD modes**
- **POD modes individually satisfy incompressibility, BCs**
- **optimality: for a given number of modes energy captured by POD basis $>$ any other basis**
  - eigenvalue $\lambda$ represents twice average KE in POD mode
  - subspace spanned by the modes corresponding to the largest eigenvalues captures the most energetic disturbances

# Low-dimensional modelling

## *POD applied to PCF - Symmetry Considerations*

- Fourier modes optimal for directions with translation symmetry

$$\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x}) = \frac{\boldsymbol{\phi}^{(n)}_{n_x n_z}(y)}{\sqrt{L_x L_z}} \exp\left(2\pi i \left(\frac{n_x x}{L_x} + \frac{n_z z}{L_z}\right)\right)$$

- enlarge set of snapshots with discrete symmetries

$$\begin{aligned}
\mathcal{P} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] &= (-u_1, -u_2, -u_3, p)(-x, -y, -z, t) \\
\mathcal{R} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] &= (u_1, -u_2, u_3, p)(x, -y, z, t) \\
\mathcal{RP} \cdot [(u_1, u_2, u_3, p)(x, y, z, t)] &= (-u_1, u_2, -u_3, p)(-x, y, -z, t)
\end{aligned}$$

- P is a point rotation about the centre of the system
- R is a reflection in the central streamwise-wallnormal $(x - y)$ plane $(z = 0)$

# Galerkin Projection

## *Galerkin projection*

- write evolution PDE as

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u})$$

- expand

$$\mathbf{u}(\mathbf{x}, t) = \sum_{n=1}^{\infty} \sum_{n_x=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} a_{n_x n_z}^{(n)}(t) \mathbf{\Phi}_{n_x n_z}^{(n)}(\mathbf{x}), \qquad a_{n_x n_z}^{(n)} \in C$$

- substitute into evolution PDE

$$\sum_{n, n_x, n_z} \dot{a}_{n_x n_z}^{(n)}(t) \mathbf{\Phi}_{n_x n_z}^{(n)}(x) = \mathbf{F}\left( \sum_{n, n_x, n_z} a_{n_x n_z}^{(n)}(t) \mathbf{\Phi}_{n_x n_z}^{(n)}(\mathbf{x}) \right)$$

- take inner product with $\mathbf{\Phi}_{m_x m_z}^{(m)}(\mathbf{x})$, use orthonormality

$$\Rightarrow \dot{a}_{m_x m_z}^{(m)}(t) = \left( \mathbf{F}\left( \sum_{n, n_x, n_z}^{\infty} a_{n_x n_z}^{(n)}(t) \mathbf{\Phi}_{n_x n_z}^{(n)}(\mathbf{x}) \right), \mathbf{\Phi}_{m_x m_z}^{(m)}(\mathbf{x}) \right)$$

# Minimal flow unit: definition

... the smallest domain in which turbulence can be sustained

## *Important references*

- **channel flow: Jiminez and Moin (JFM, 1991)**
  - found spanwise width $\approx 100$ wall units required to sustain turb.
    - cf. Kline et al, JFM 1967; $100 \approx$ streak spacing in turb.boundary layer
- **PCF: Hamilton et al (JFM, 1995) - "HKW"**
  - found analogous minimal flow unit for PCF
  - at $Re = 400$ domain size was $L_x \times L_z = 1.75\pi \times 1.2\pi$
  - suggested periodic orbit $\approx$ "backbone" for turbulence
  - insipired several low-d models (Waleffe, Phys. Fluids 1995/7)
- **PCF: Kawahara & Kida (JFM, 2001) -"K & K"**
  - examined minimal flow unit as defined by HKW
  - discovered an *unstable* periodic orbit buried inside DNS

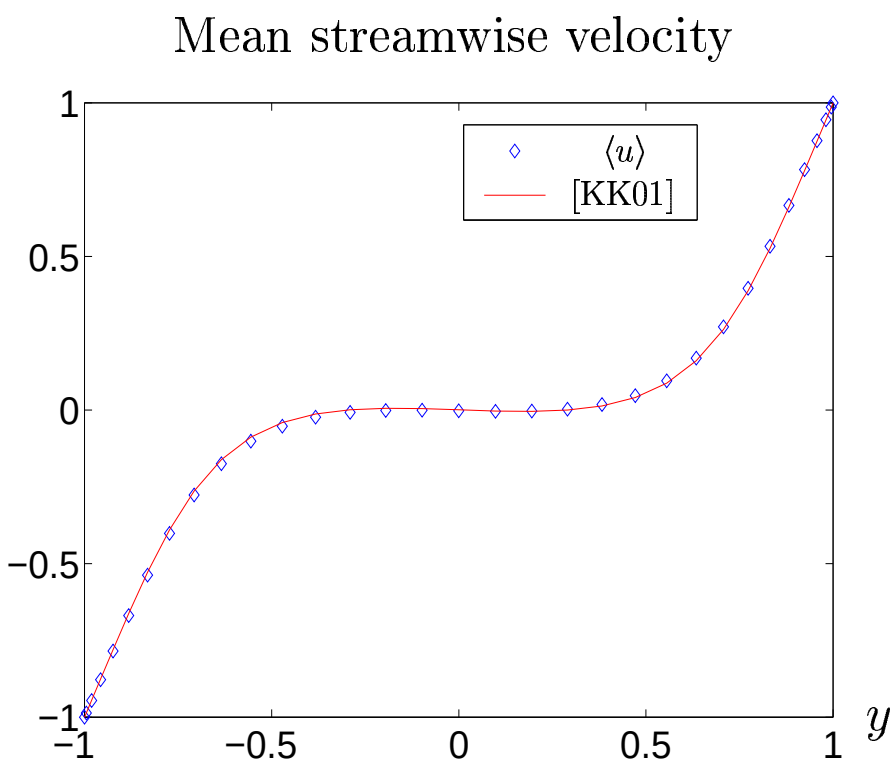
# Minimal flow unit: simulation

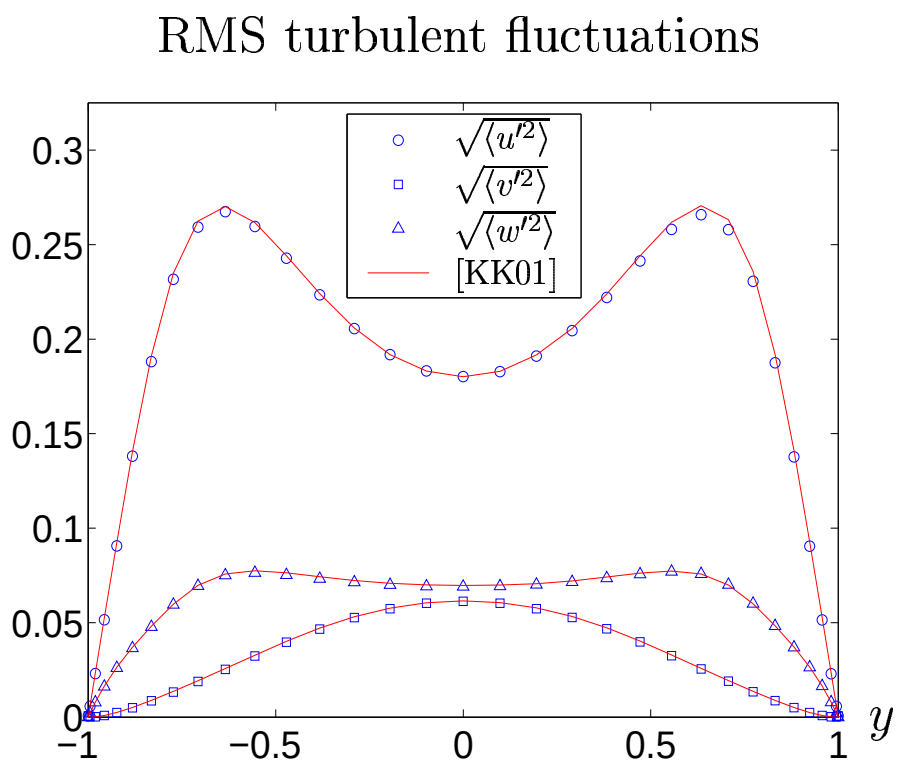
## *Numerical database*

- **basic channel code provided by C. Rowley**
- **algorithm: Kim, Moin & Moser, JFM 1987**
  - pseudospectral spatial discretisation
  - Crank-Nicholson routine for linear terms
  - 3rd Order Runge-Kutta for nonlinear terms
- **grid:** $N_x = N_z = 16$ **Fourier moves**
  **33 Chebyshev Polynomials in** $y$ **direction**
  **(cf. HKW, K & K)**
- **sustained turb. at** $Re = 400$ **difficult to achieve**
  - begin with random state at $Re = 625$
  - ran until converged, reduced $Re$ and repeated
- **integrated time = 20,000 outer time units**
- **one sample saved every 5 time units (4000 total)**

# Minimal flow unit: simulation

## *Turbulent statistics comparisons*

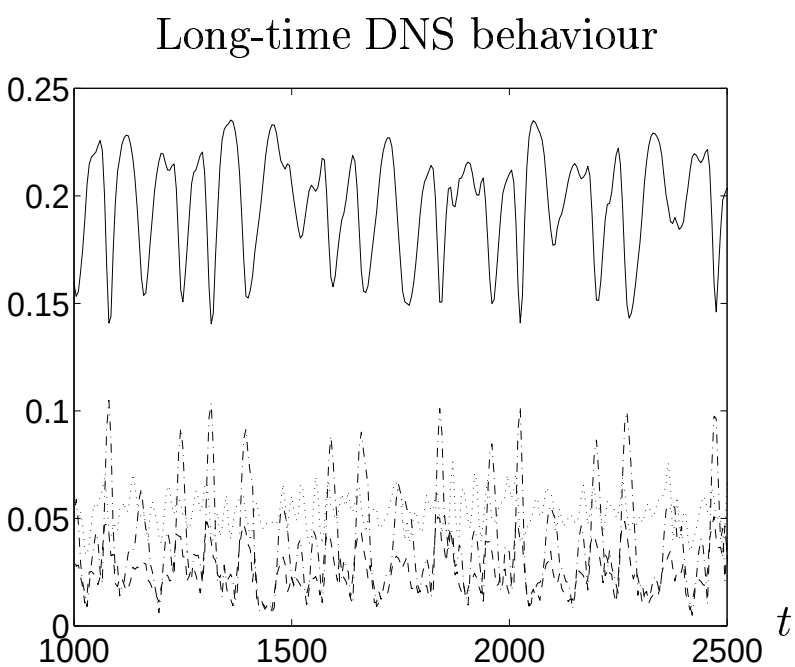


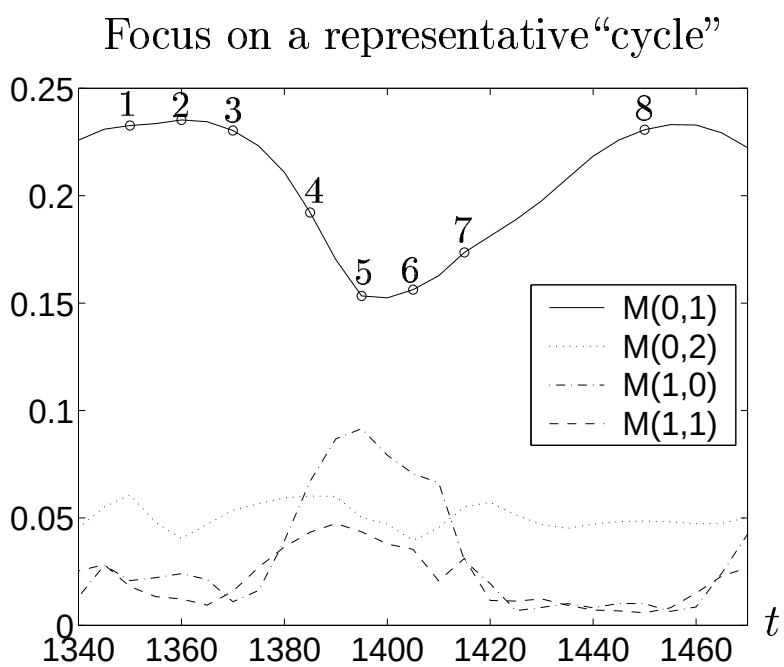



# Minimal flow unit: phenomenology

- For PCF in the MFU $\exists$ a "self-sustaining process"
  - **define modal RMS velocity as**

$$\mathbf{M}(n_x, n_z) \stackrel{\text{def}}{=} \left( \int_{-1}^{1} \left[ \tilde{u}_1^2(n_x, y, n_z) + \tilde{u}_2^2(n_x, y, n_z) + \tilde{u}_3^2(n_x, y, n_z) \right] dy \right)^{1/2}$$



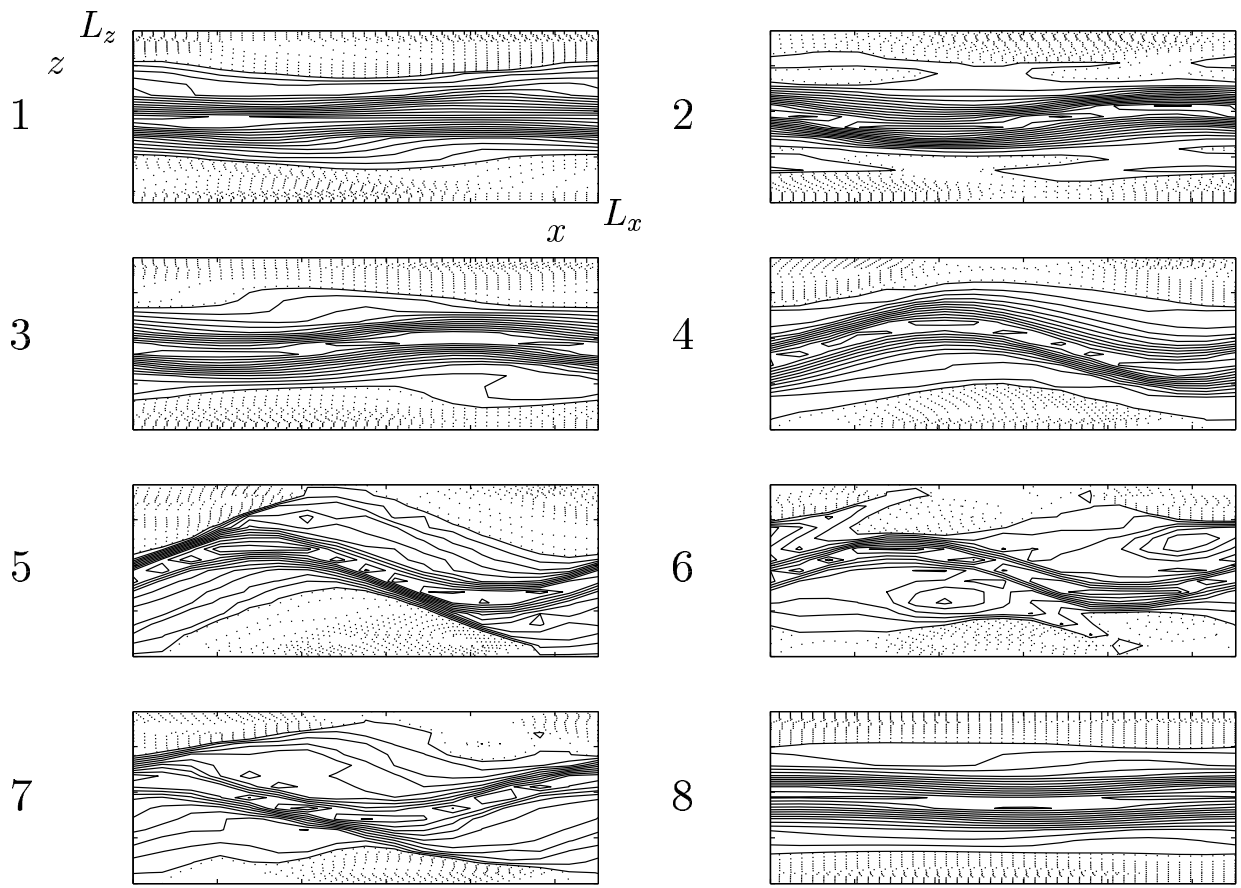




Note flow is roughly periodic - with period $80 \approx 100$ (cf. HKW Fig 3-a)

# Minimal flow unit: phenomenology

- The process of streak breakdown

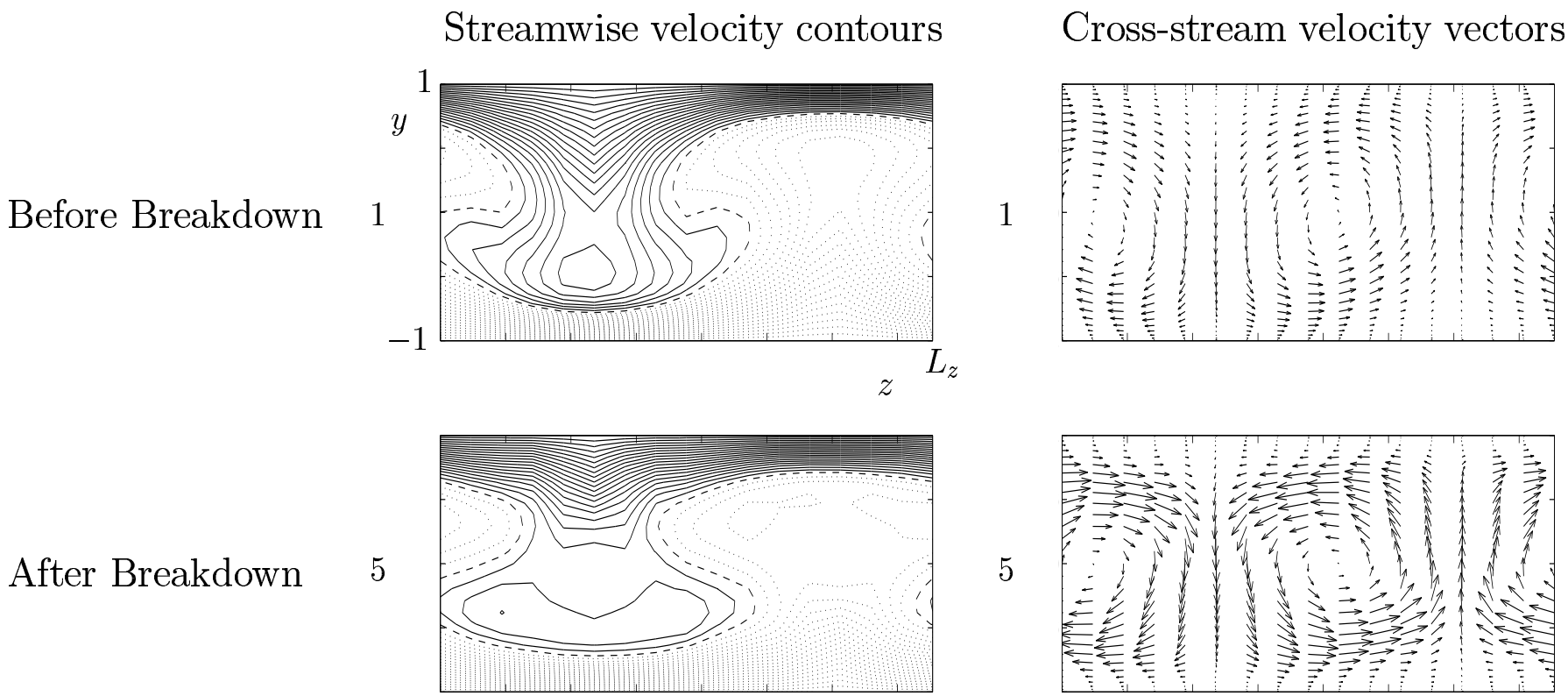
Midplane $x$ velocity contours - top view




(cf. HKW Fig. 2)

# Minimal flow unit: phenomenology

- The $x$-independent modes & streak breakdown




N.B. plots on the left include the laminar profile (cf. HKW Fig. 4)

# Minimal flow unit: objectives

- PCF-MFU: a test-bed for low-d models
  - **"low-order" tests**
    - approximate reproduction of energy budgets
    - evolution of model amplitudes $\sim$ evolution of DNS amplitudes
    - approximate reproduction of time-scales
  - **"high-order" tests**
    - model velocity field reconstructions similar to DNS
    - maintain important properties of flow being considered
      eg. for PCF, stability of the laminar state
    - admits a plausible bifurcation scenario - transition to turbulence

# Minimal flow unit: analysis

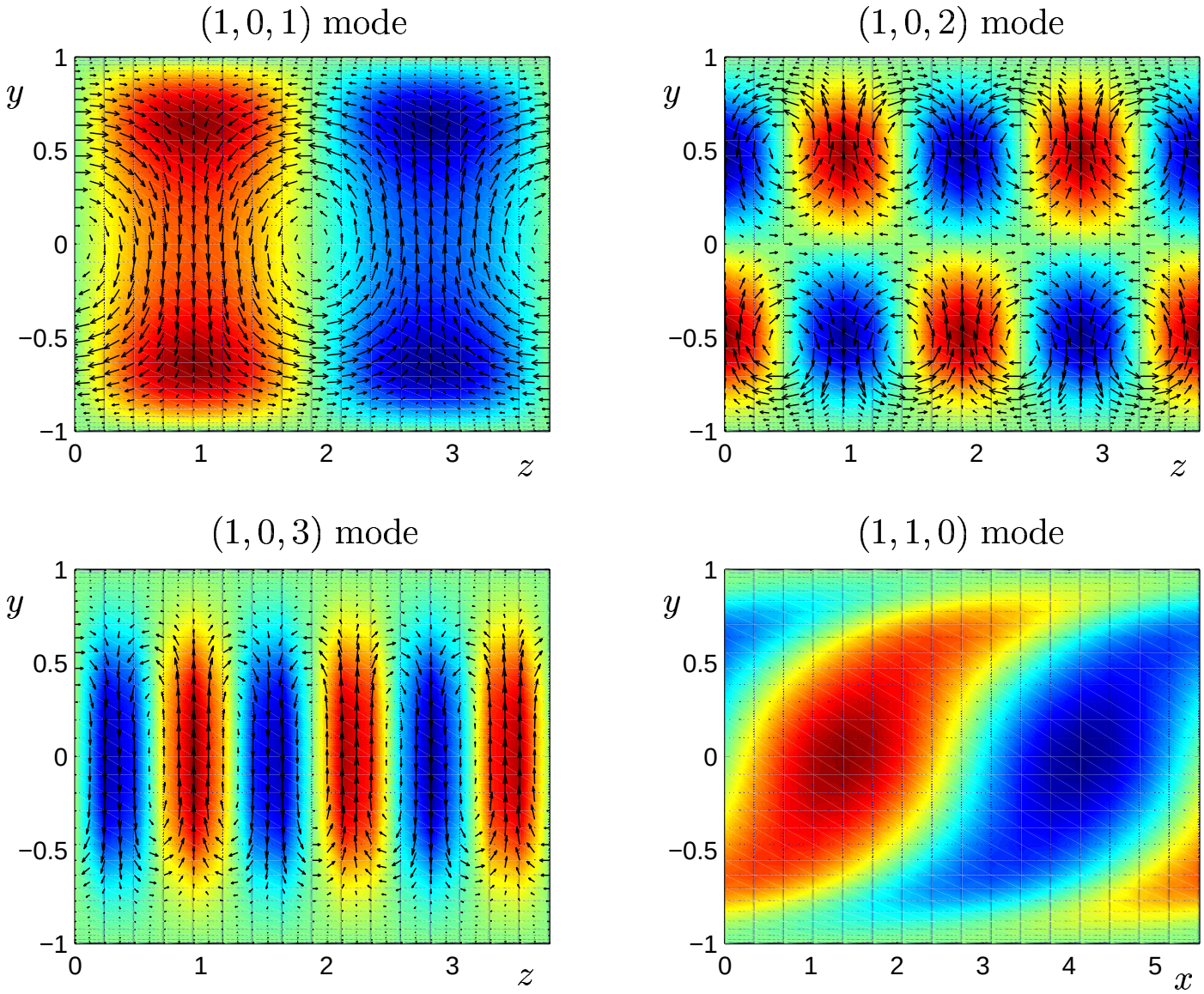
- POD decomposition of PCF-MFU

| $(n, n_x, n_z)$ | $\lambda^{(n)}_{n_x,n_z}$ | $\%E^{(n)}_{n_x,n_z}$ | Model |
|---|---|---|---|
| $(1, 0, 0)$ | 4.4550 | 68.02 | ☐ |
| $(1, 0, \pm 1)$ | 0.7821 | 23.88 | ☐ |
| $(1, 0, \pm 2)$ | 0.0543 | 1.66 | ☐ |
| $(1, \pm 1, 0)$ | 0.0386 | 1.18 | ☐ |
| $(1, 0, \pm 3)$ | 0.0195 | 0.59 | |
| $(2, 0, 0)$ | 0.0174 | 0.27 | "exclude n=2" |
| $(2, 0, \pm 1)$ | 0.0123 | 0.38 | "exclude n=2" |
| $(1, \pm 1, \pm 2)$ | 0.0109 | 0.33 | |
| $(1, \pm 1, \pm 1)$ | 0.0090 | 0.27 | ☐ |
| ... | | | |

- choose $(1, \pm 1, \pm 1)$ instead of $(1, \pm 1, \pm 2)$ since the former appears to be more important in the streak formation/breakdown process

# Minimal flow unit: analysis

- Some of the POD modes



# Minimal flow unit: analysis

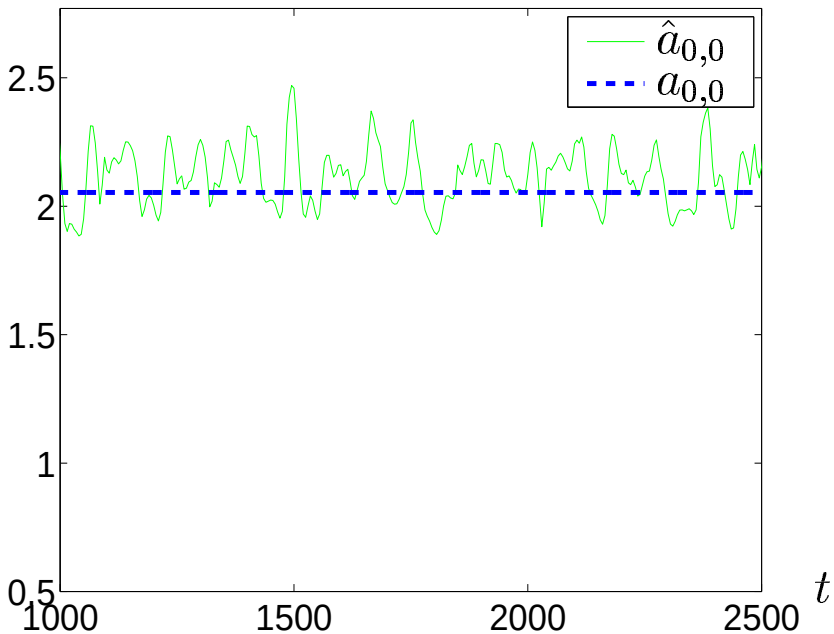
- **Constructing a model from the modes** $(n, n_x, n_z) = (1,0,0), (1,0,1), (1,0,2), (1,1,0), (1,1,\pm 1)$ $\rightarrow$ **a set of 6 ODEs (1 real + 5 complex) of the form**

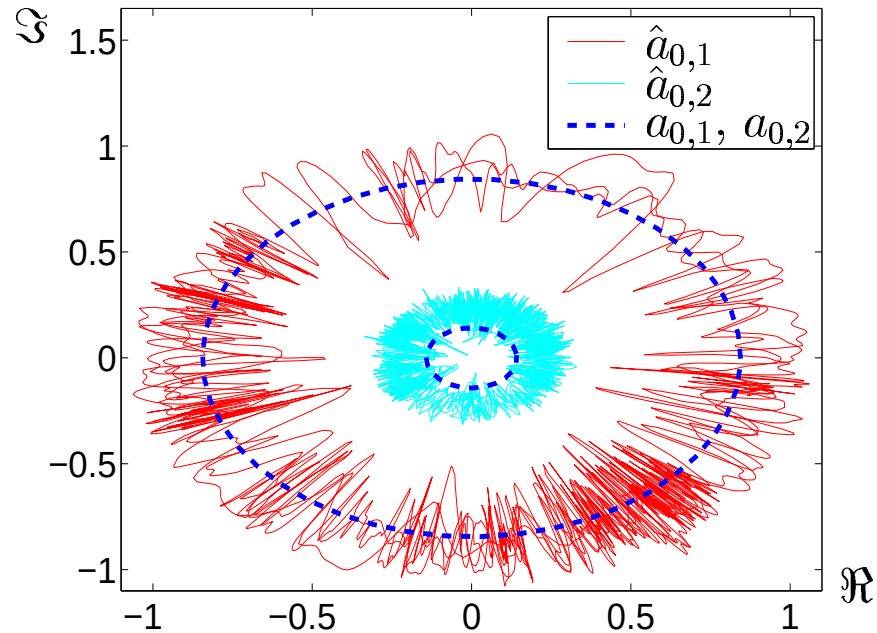
$$
\begin{aligned}
\dot{a}^{(1)}_{0,0} &= -A^{(1)}_{0,0} a^{(1)}_{0,0} + \sum B^{(1)}_{n_x,n_z} |a^{(1)}_{n_x,n_z}|^2 \\
\dot{a}^{(1)}_{n_x,n_z} &= \left(A^{(1)}_{n_x,n_z} - B^{(1)}_{n_x,n_z} a^{(1)}_{0,0}\right) a^{(1)}_{n_x,n_z} + N^{(1)}_{n_x,n_z} \\
N^{(1)}_{n_x,n_z} &\stackrel{\text{def}}{=} C_{n_x,n_z m_x m_z} a^{(1)}_{m_x,m_z} a^{(1)}_{n_x-m_x,n_z-m_z}
\end{aligned}
$$

- **only keep one "quantum" number per Fourier pair** $\Longrightarrow$ **no interaction at the linear level**
- **The 3 ODEs corresponding to interaction between** $(n, n_x, n_z) = (1,0,0), (1,0,1), (1,0,2)$ **modes comprises the 0:1:2 resonance (article in preparation)**
- **seek to compare the ODE models to the DNS; projection of DNS onto** $(n, n_x, n_z)$ **denoted by** $\hat{a}^{(n)}_{n_x,n_z}$

# Minimal flow unit: coupled models

- Projection of DNS onto 3 most energetic POD modes

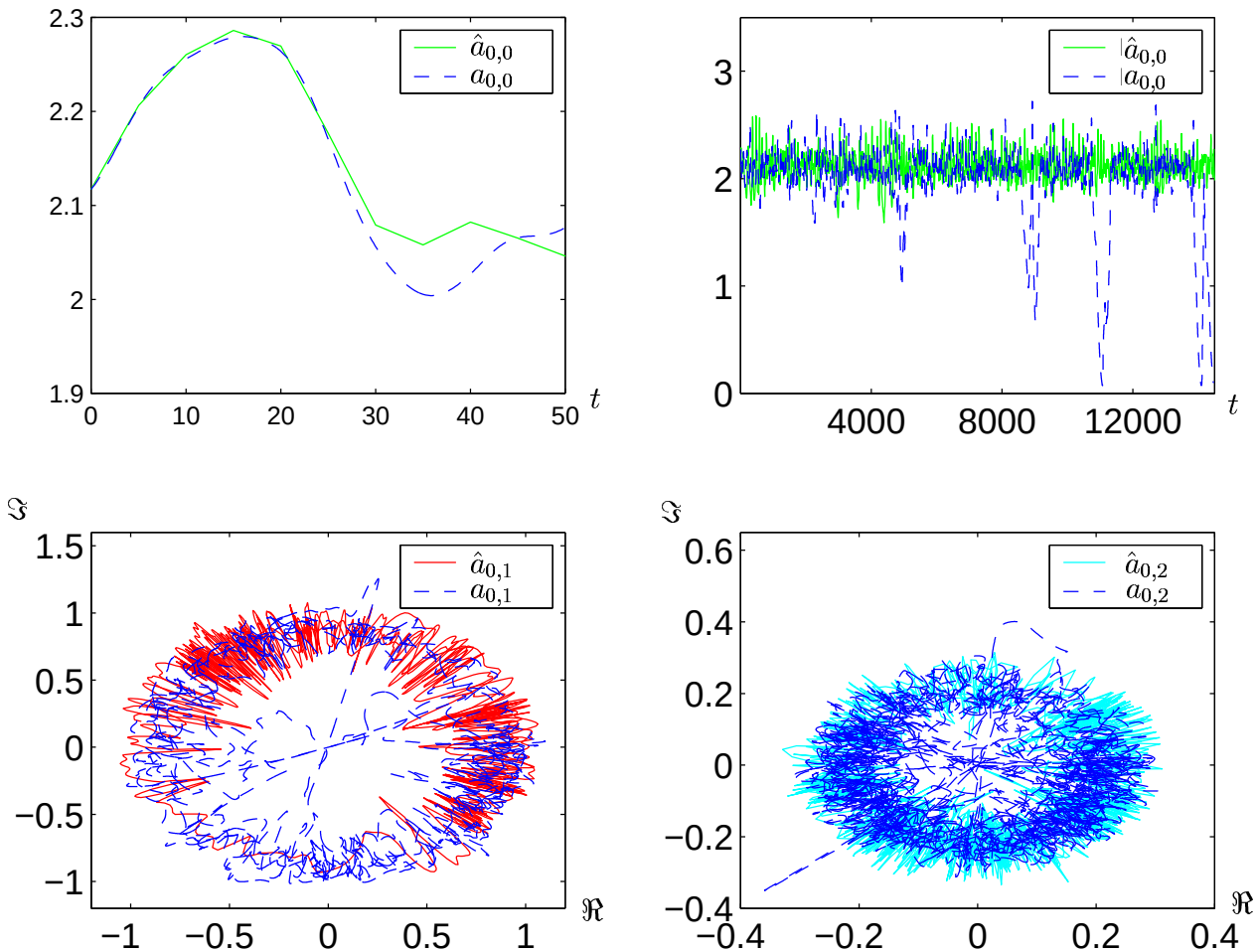



- $(1,0,1)$, $(1,0,2)$ move around a circle in a slow, chaotic fashion - relatively fast motion in the radial direction
- $(1,1,0)$, $(1,1,\pm 1)$ more complex, of less energetic importance
- model caricatures this complex behaviour with a simple travelling wave

# Minimal flow unit: coupled models

- How many modes would be required for accurate reproduction of DNS?
  - $\approx 1000$, **as these 600 mode projections indicate...**



# Minimal flow unit: coupled models

## *Effect of neglected terms*

- **Terms neglected through truncation which are *small on average* have a significant effect**
- **Exact equations for the evolution of the modal amplitudes are of form**

$$\dot{a}^{(n)}_{n_x n_z}(t) = \text{set of retained terms} + T^{(n)}_{n_x n_z}$$

- **Simplest possible approximation is of the form**

$$T^{(n)}_{n_x n_z} = -\beta^{(n)}_{n_x n_z} a^{(n)}_{n_x n_z}$$

**(Heisenberg Eddy-viscosity model)**

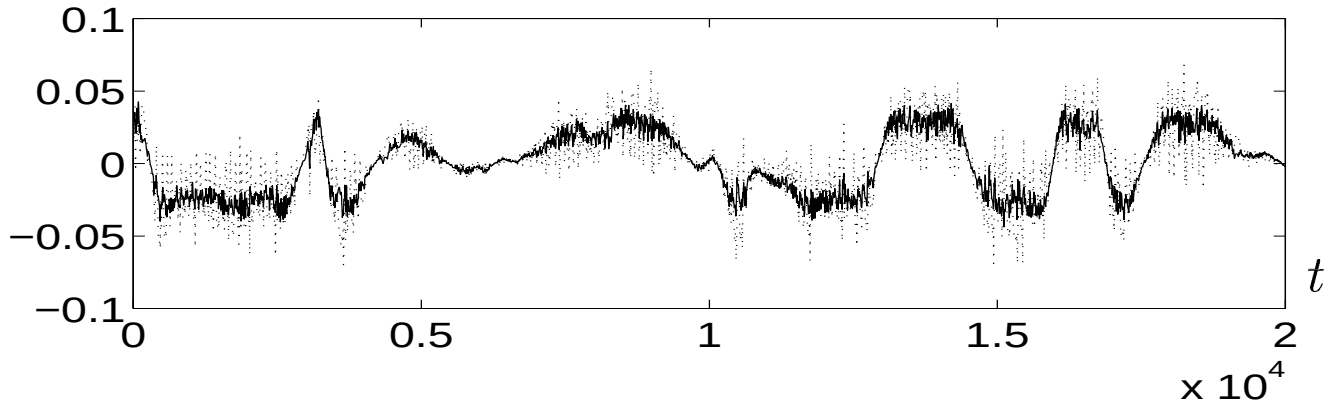
- **choose $\beta^{(n)}_{n_x,n_z}$ through least-squares fit**

# Minimal flow unit: coupled models

- How well does this approximation work?
  - **For some modes, it works very well...**
    - eg. the (1,0,2) mode

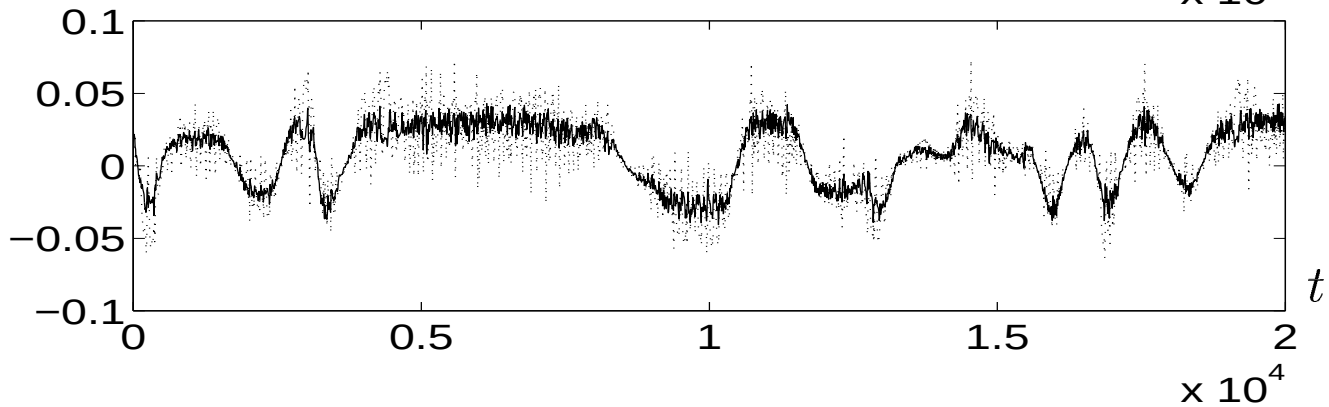
dotted: $\Re(T_{0,2}^{(1)})$

solid: $\Re(-\beta_{0,2}^{(1)}\hat{a}_{0,2}^{(1)})$




dotted: $\Im(T_{0,2}^{(1)})$

solid: $\Im(-\beta_{0,2}^{(1)}\hat{a}_{0,2}^{(1)})$

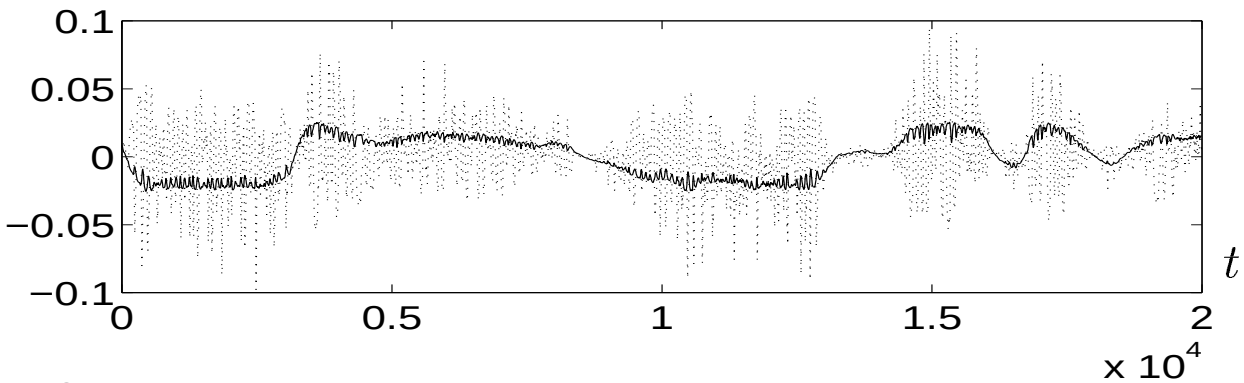


# Minimal flow unit: coupled models

- **For others it does not work very well at all...**
  - eg. the (1,0,1) mode

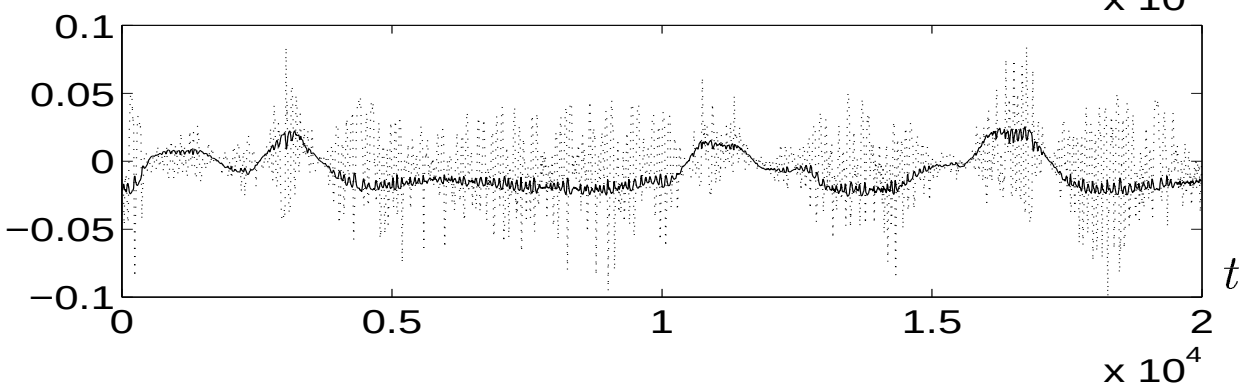
dotted: $\Re(T_{0,1}^{(1)})$

solid: $\Re(-10\beta_{0,1}^{(1)}\hat{a}_{0,1}^{(1)})$




dotted: $\Im(T_{0,1}^{(1)})$

solid: $\Im(-10\beta_{0,1}^{(1)}\hat{a}_{0,1}^{(1)})$



# Minimal flow unit: coupled models

- **attempt to average out this variability with a spectral transfer model**

$$\dot{a}^{(n)}_{n_x,n_z} = \cdots - \alpha\nu(n_x^2 + n_z^2)a^{(n)}_{n_x,n_z}$$

**with**

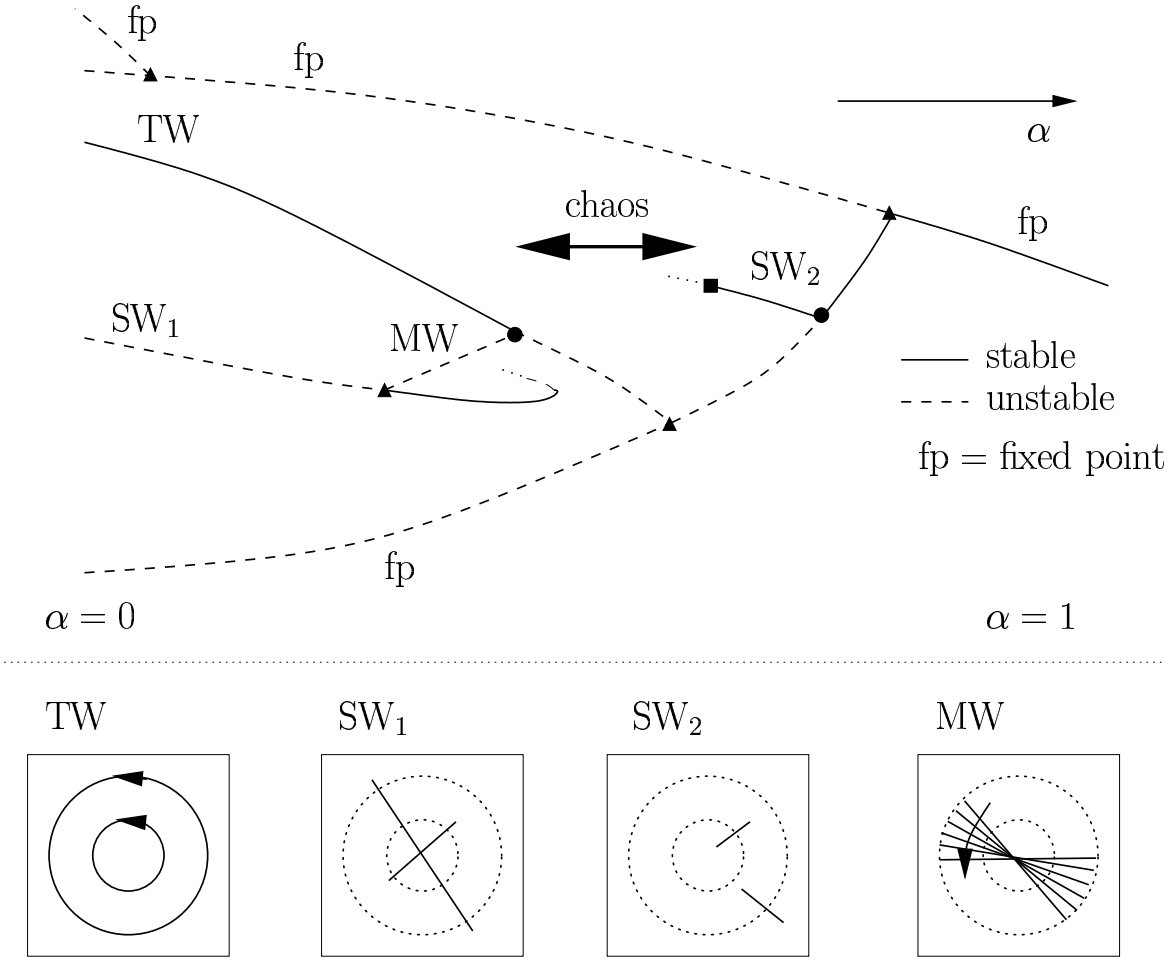
$\nu = 0.0333$ **(computed from weighted average of $\beta$s)**
$\alpha =$ an $\mathcal{O}(1)$ *bifurcation parameter*

  - Note: no term added to the equation for $\dot{a}^{(1)}_{0,0}$

- **fix $Re = 400$, examine behavior as $\alpha$ varies**

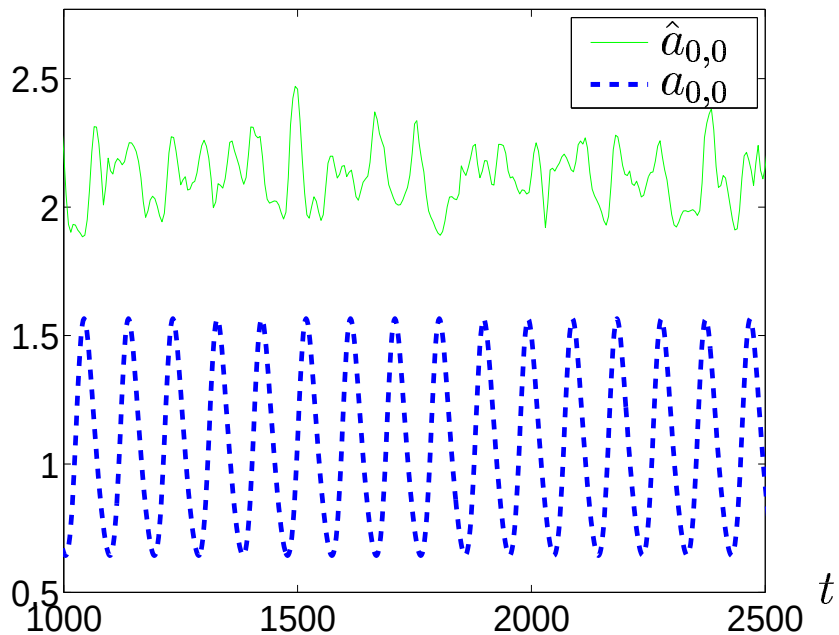
# Minimal flow unit: coupled models

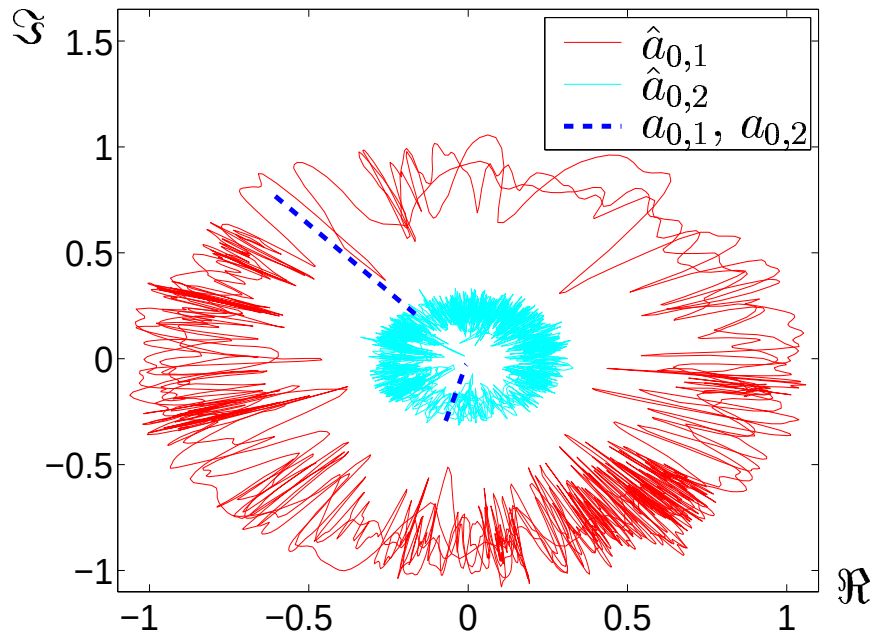
- Bifurcation diagram



# Minimal flow unit: coupled models

- DNS vs. model at $Re = 400$, $\alpha = 0.8$



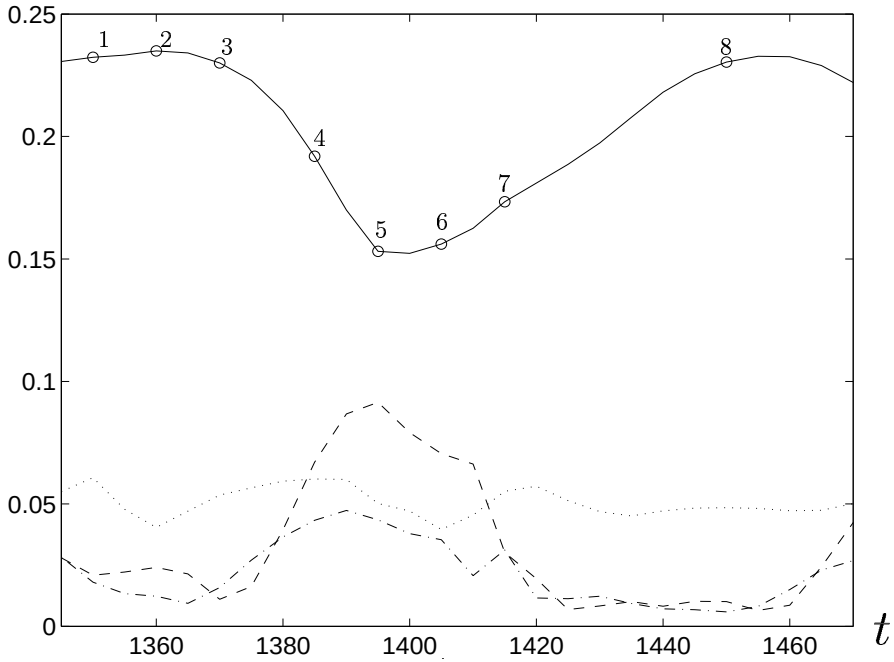




- instead of travelling waves now have *standing* waves
- amplitude of $a_{0,0}^{(1)}$ far too low, others aren't too bad

# Minimal flow unit: coupled models

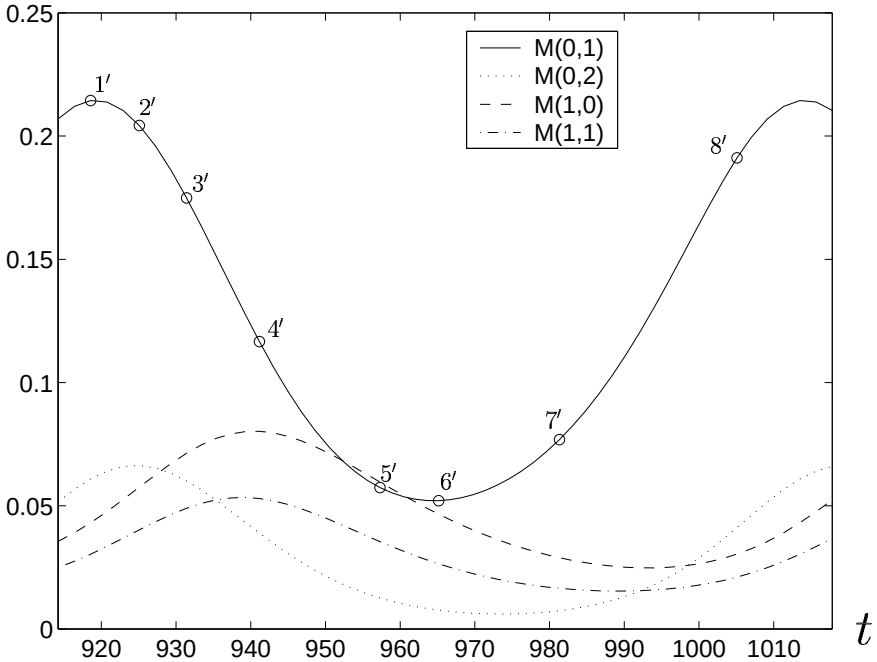
- DNS vs. model RMS modal velocities

Representative cycle from the DNS




Period $\approx 80 - 100$ outer time units

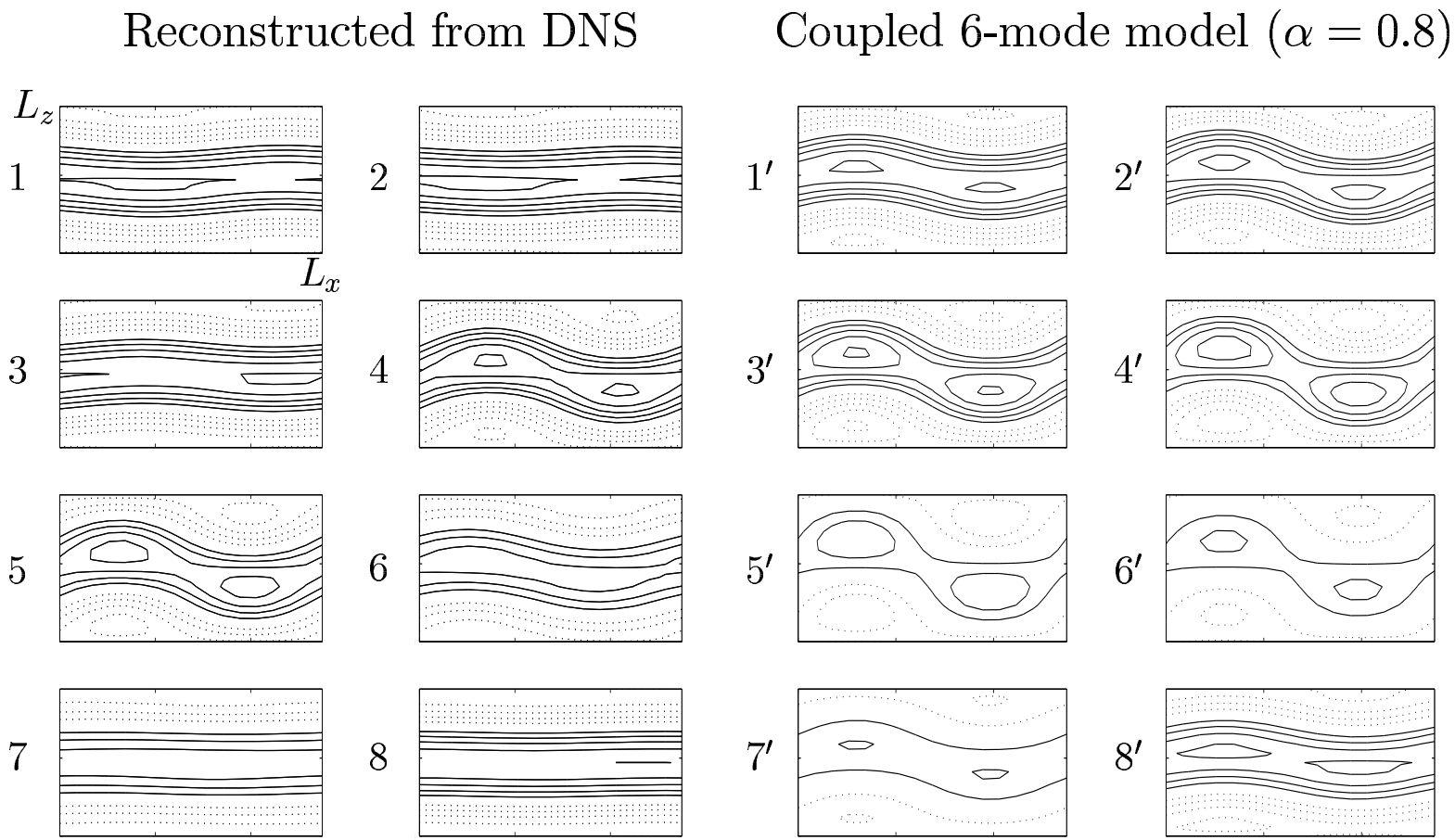
Coupled 6-mode model ($\alpha = 0.8$)




Period $\approx 95$ outer time units

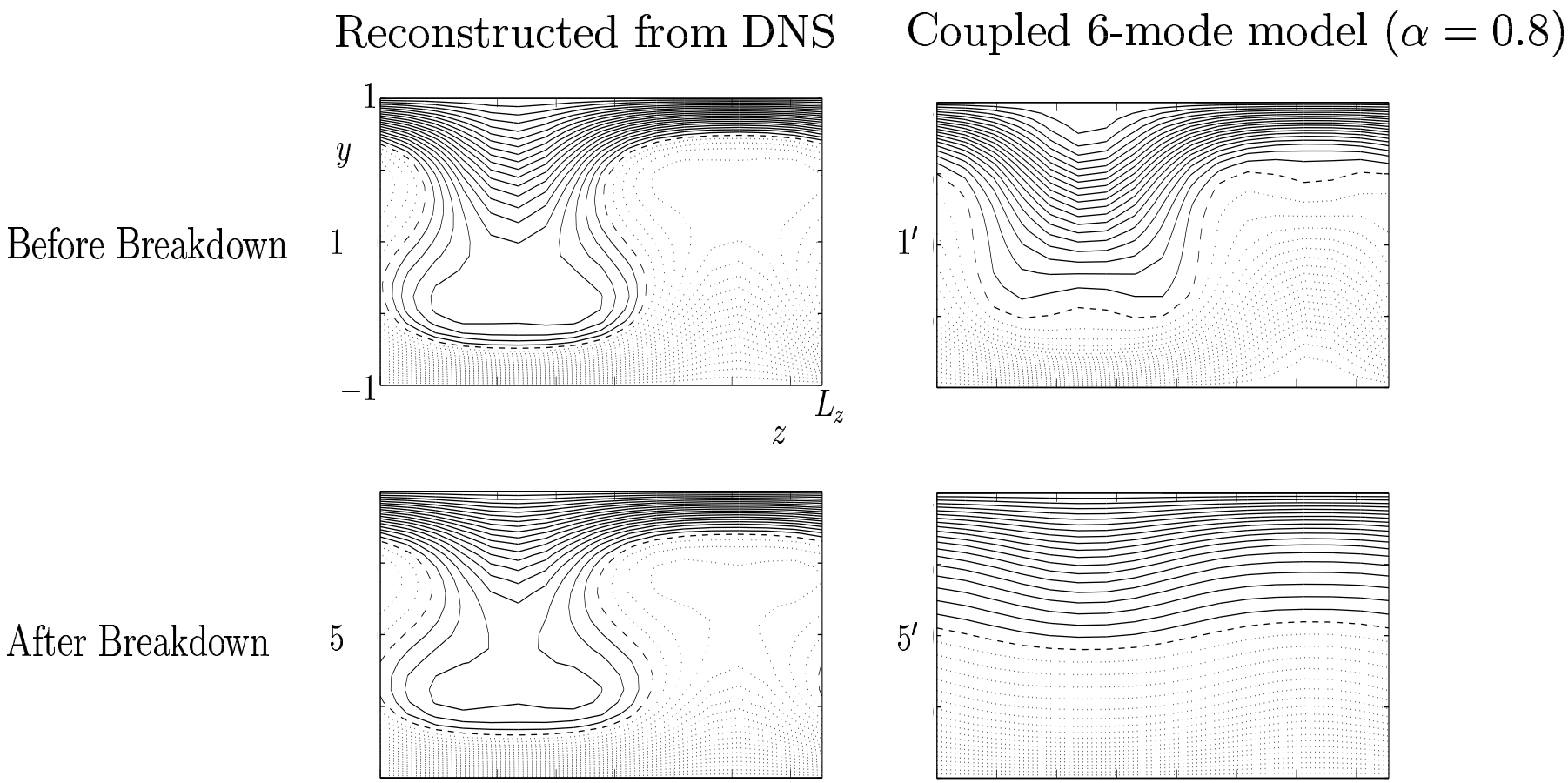
# Minimal flow unit: coupled models

- DNS vs. model streak-breakdown process



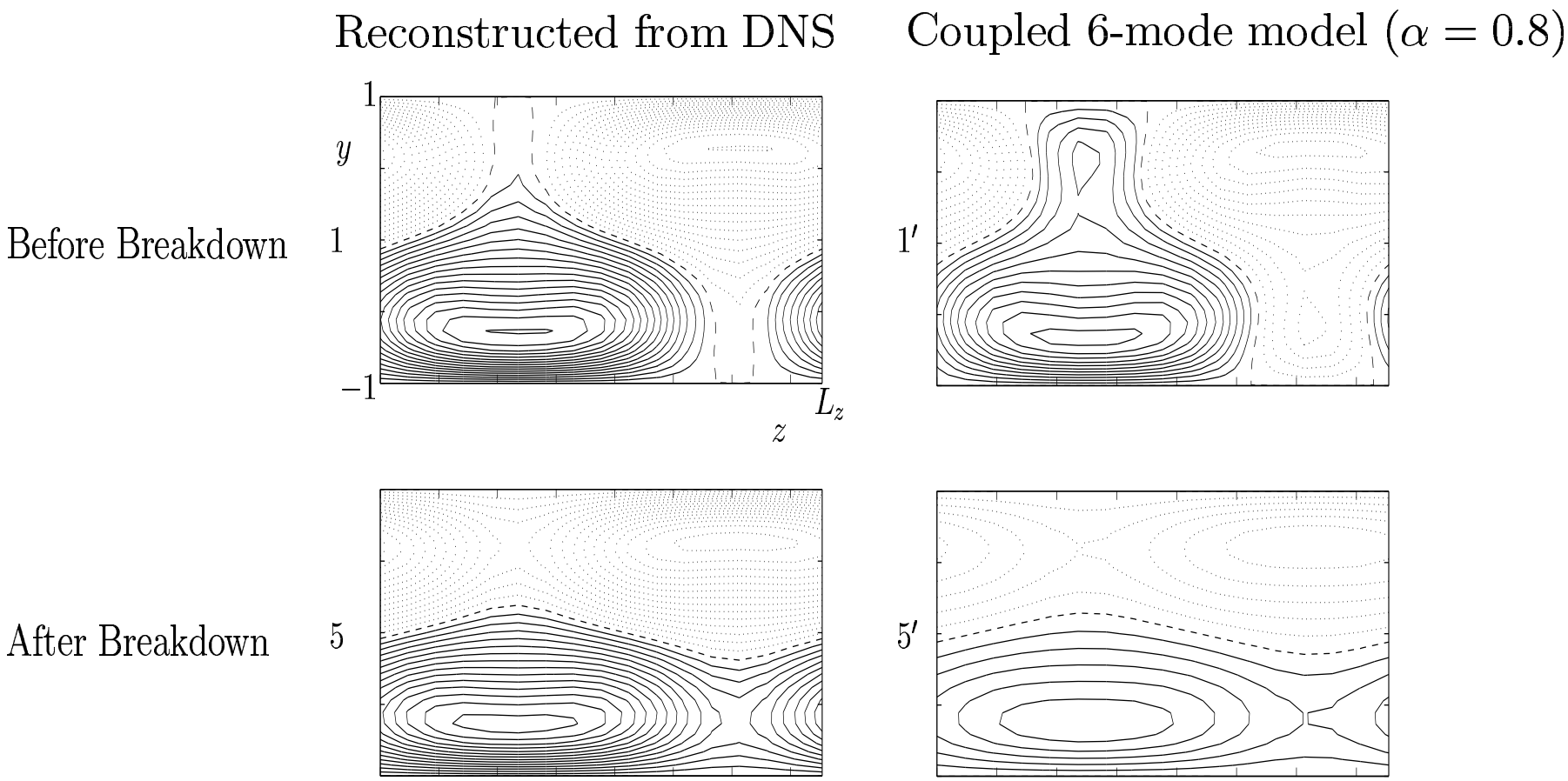
# Minimal flow unit: coupled models

- DNS vs. model streamwise velocity contours (including laminar state)



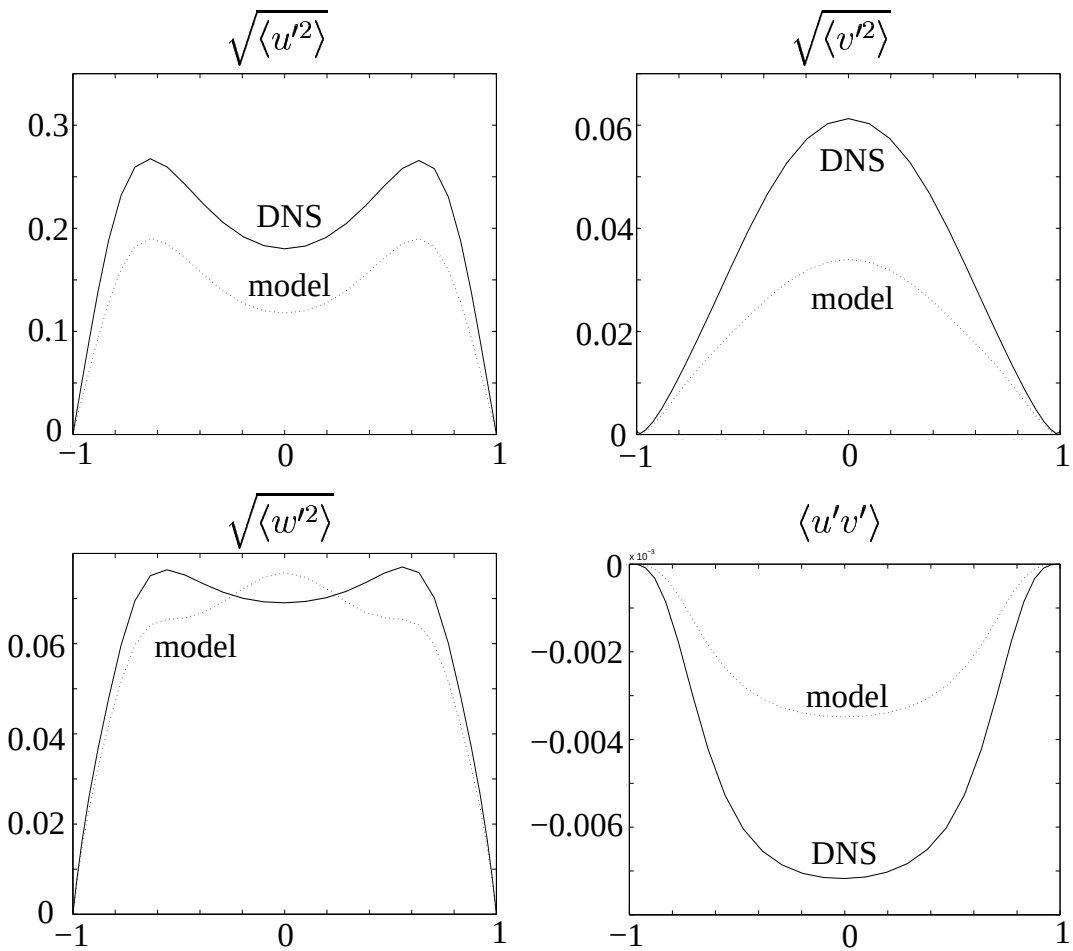
# Minimal flow unit: coupled models

- DNS vs. model streamwise velocity contours (neglecting laminar state)



# Minimal flow unit: coupled models

- DNS vs. 6-mode model turbulent statistics



# Minimal flow unit: uncoupled models

- Model captures MFU turbulence as periodic orbit
  - **approximately correct time-scales, phase relationships, and velocity reconstructions**
- But, predicts laminar state becomes unstable
  $\Rightarrow$ uncoupled expansion  Ref: F. Waleffe, *Phys. Fluids* **7**:3060, 1995

$$\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x}) = \underbrace{\mathbf{\Phi}^{(n)[1]}_{n_x n_z}(\mathbf{x})}_{P\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x})} + \underbrace{\mathbf{\Phi}^{(n)[2]}_{n_x n_z}(\mathbf{x})}_{(I-P)\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x})}$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{n=1}^{\infty}\sum_{n_x=-\infty}^{\infty}\sum_{n_z=-\infty}^{\infty}\sum_{m=1}^{2} b^{(n)[m]}_{n_x n_z}(t)\mathbf{\Phi}^{(n)[m]}_{n_x n_z}$$

- first suggested to avoid paradoxical behaviour of ODE models based upon *streamwise-invariant* POD modes

# Minimal flow unit: uncoupled models

- Streamwise-invariant $\Longrightarrow$ perturbations must decay!
  - for mean velocity $U$, peturbations $u, v, w$ and $\partial/\partial x = 0$ we have

$$D/Dt = \partial/\partial t + u_2 \partial/\partial y + u_3 \partial/\partial z \qquad \text{(convective derivative)}$$

$$\frac{D}{Dt}(U + u_1) = \frac{1}{Re}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(U + u_1) \quad \text{(NS Equation)}$$

$$\frac{d}{dt}\int\int (u_2^2 + u_3^2)\, dydz = -2\nu \left\{\int\int \omega_x^2 dydz\right\} \quad (\omega_x = x\text{-vorticity})$$

$\Rightarrow$ all disturbances must decay

  - "artificial coupling" $\Rightarrow$ streamwise-invariant model sustained dynamics
  - Berkooz, Holmes & Lumley (JFM 1991) proposed to remove coupling
  - later generalised by Waleffe (Phys. Fluids 1997)

# Minimal flow unit: uncoupled models

- Uncoupled basis functions

$$\text{streamwise-invariant:}\quad \boldsymbol{\phi}^{(n)[1]}_{0,n_z}(y) = \begin{pmatrix} \boldsymbol{\phi}^{(n)}_{1,0,n_z}(y) \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\phi}^{(n)[2]}_{0,n_z}(y) = \begin{pmatrix} 0 \\ \boldsymbol{\phi}^{(n)}_{2,0,n_z}(y) \\ \boldsymbol{\phi}^{(n)}_{3,0,n_z}(y) \end{pmatrix},$$

$$\text{spanwise-invariant:}\quad \boldsymbol{\phi}^{(n)[1]}_{n_x,0}(y) = \begin{pmatrix} \boldsymbol{\phi}^{(n)}_{1,n_x,0}(y) \\ \boldsymbol{\phi}^{(n)}_{2,n_x,0}(y) \\ 0 \end{pmatrix}, \quad \boldsymbol{\phi}^{(n)[2]}_{n_x,0}(y) = \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\phi}^{(n)}_{3,n_x,0}(y) \end{pmatrix}$$

- new basis functions: pairwise orthogonal & divergence free

$$\begin{aligned} (\boldsymbol{\Phi}^{(n)[m]}_{n_x,n_z}(\mathbf{x}), \boldsymbol{\Phi}^{(n')[m']}_{n_x,n_z}(\mathbf{x})) &= e^{(n)[m]}_{n_x,n_z}\delta_{nn'}\delta_{mm'} \\ \nabla \cdot \boldsymbol{\Phi}^{(n)[m]}_{n_x,n_z}(\mathbf{x}) &= 0 \text{ for } m = 1, 2 \end{aligned}$$

- do not normalise uncoupled modes $\implies e^{(n)[m]}_{n_x,n_z} \neq 1$

  however: $e^{(n)[1]}_{n_x,n_z} + e^{(n)[2]}_{n_x,n_z} = 1$

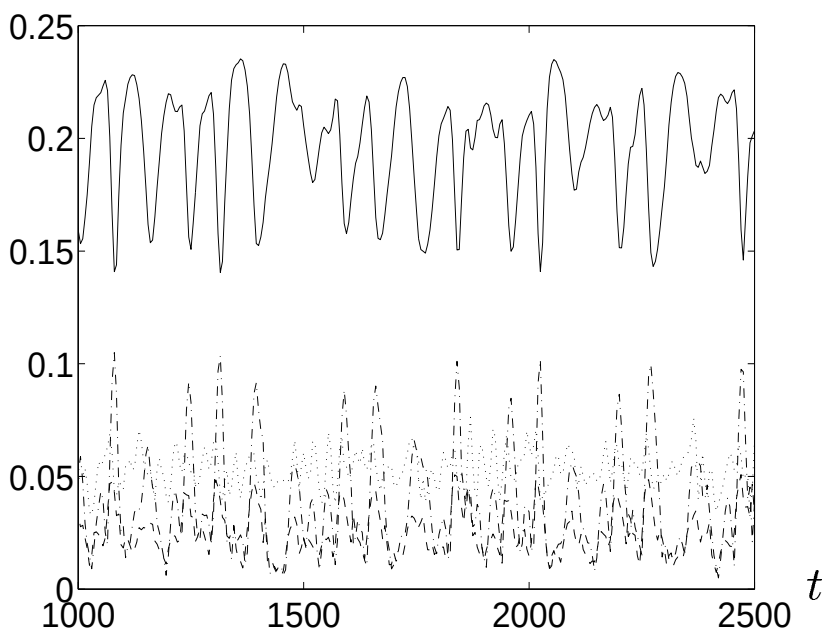
# Minimal flow unit: uncoupled models

## Uncoupled 6-mode model

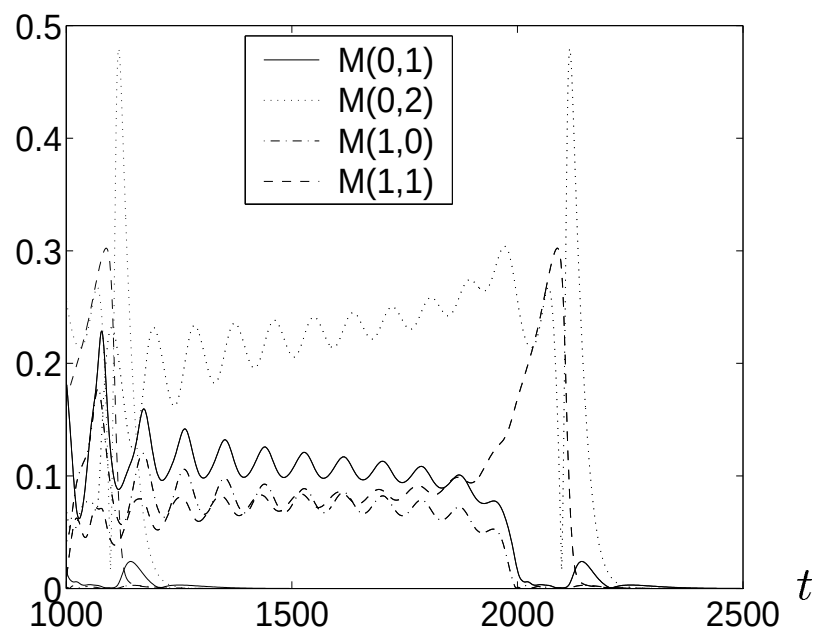
- **model is *not* streamwise invariant**
  **$\Longrightarrow$ might expect sustained behaviour**

**Plot RMS modal velocities:** $M(n_x, n_z) = \dfrac{1}{\sqrt{L_x L_z}} \sum_n |a^{(n)}_{n_x,n_z}|$

Long-time DNS behaviour



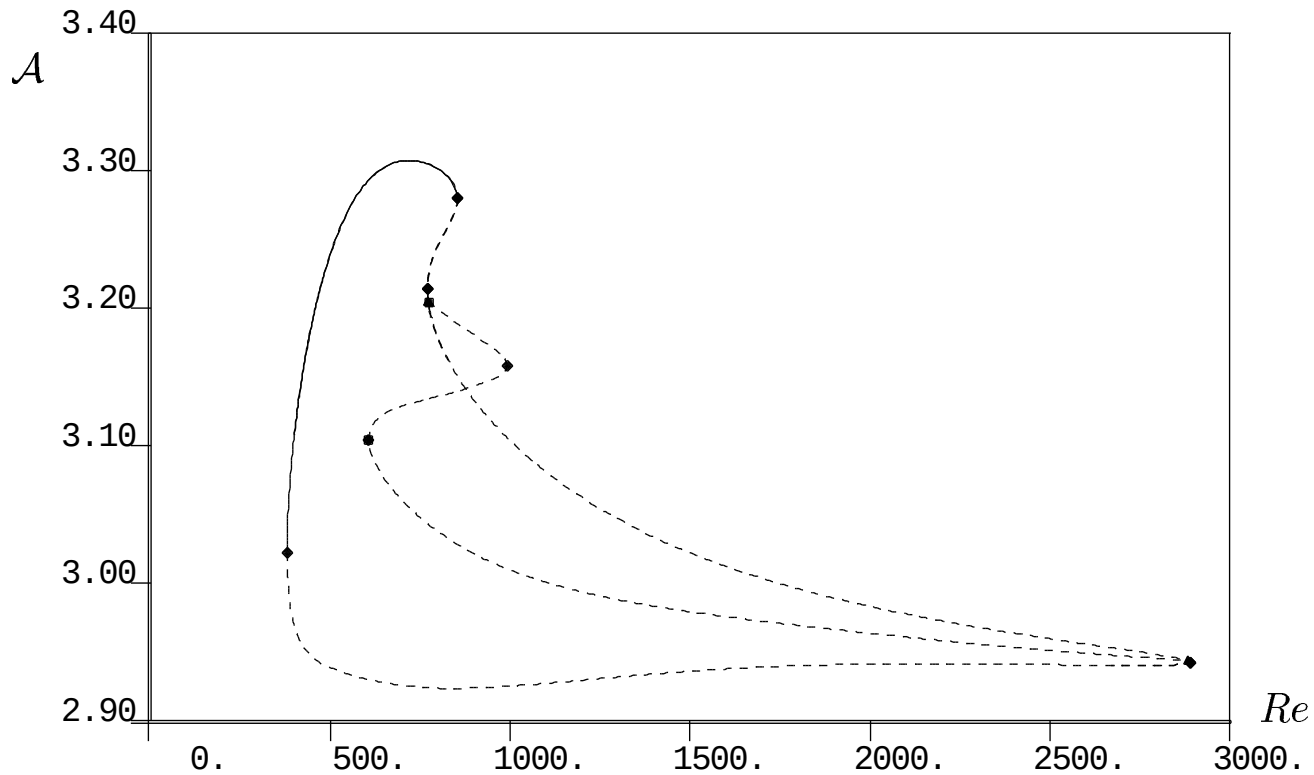

Uncoupled 6-mode model



# Minimal flow unit: uncoupled models

- Uncoupled 9-mode model
  - **Laminar solution is now *stable* with** $\alpha > 0.2179$
    - small $\|\mathbf{a}^{UC}(0)\| \rightarrow$ laminar solution as $t \rightarrow \infty$
    - larger $\|\mathbf{a}^{UC}(0)\| \rightarrow$ periodic orbit, originating in an SN bifurcation

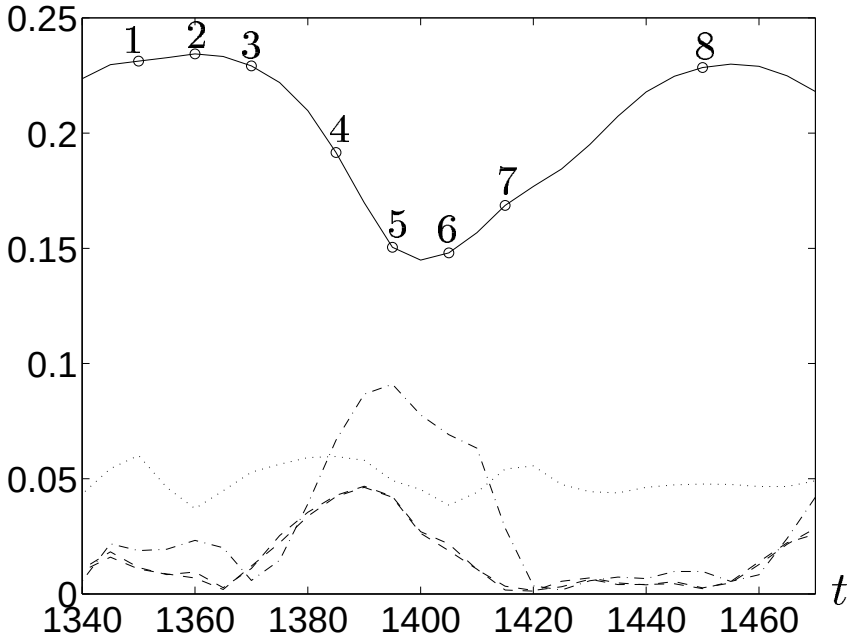
AUTO $\rightarrow$ periodic orbit orginates in a SN bifurcation, forming an isola



# Minimal flow unit: uncoupled models

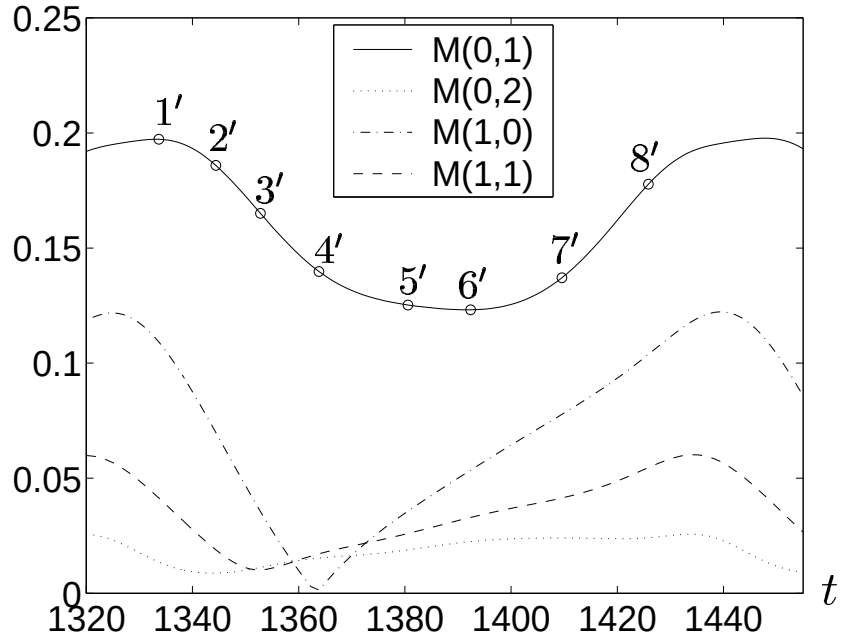
- DNS vs. model RMS modal velocities

Representative cycle from the DNS




Period $\approx 80 - 100$ outer time units

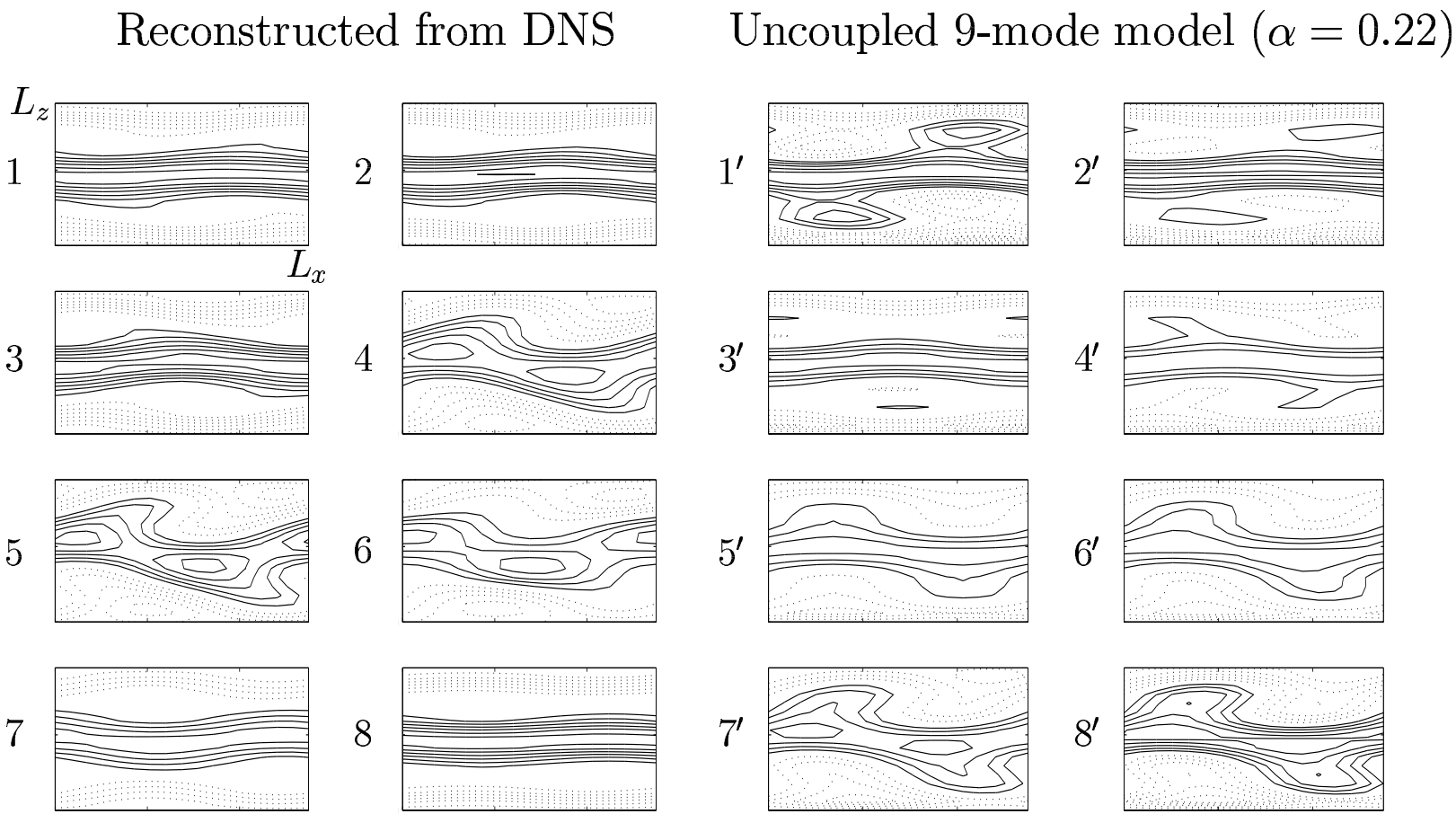
Uncoupled 9-mode model ($\alpha = 0.22$)




Period $\approx 92$ outer time units

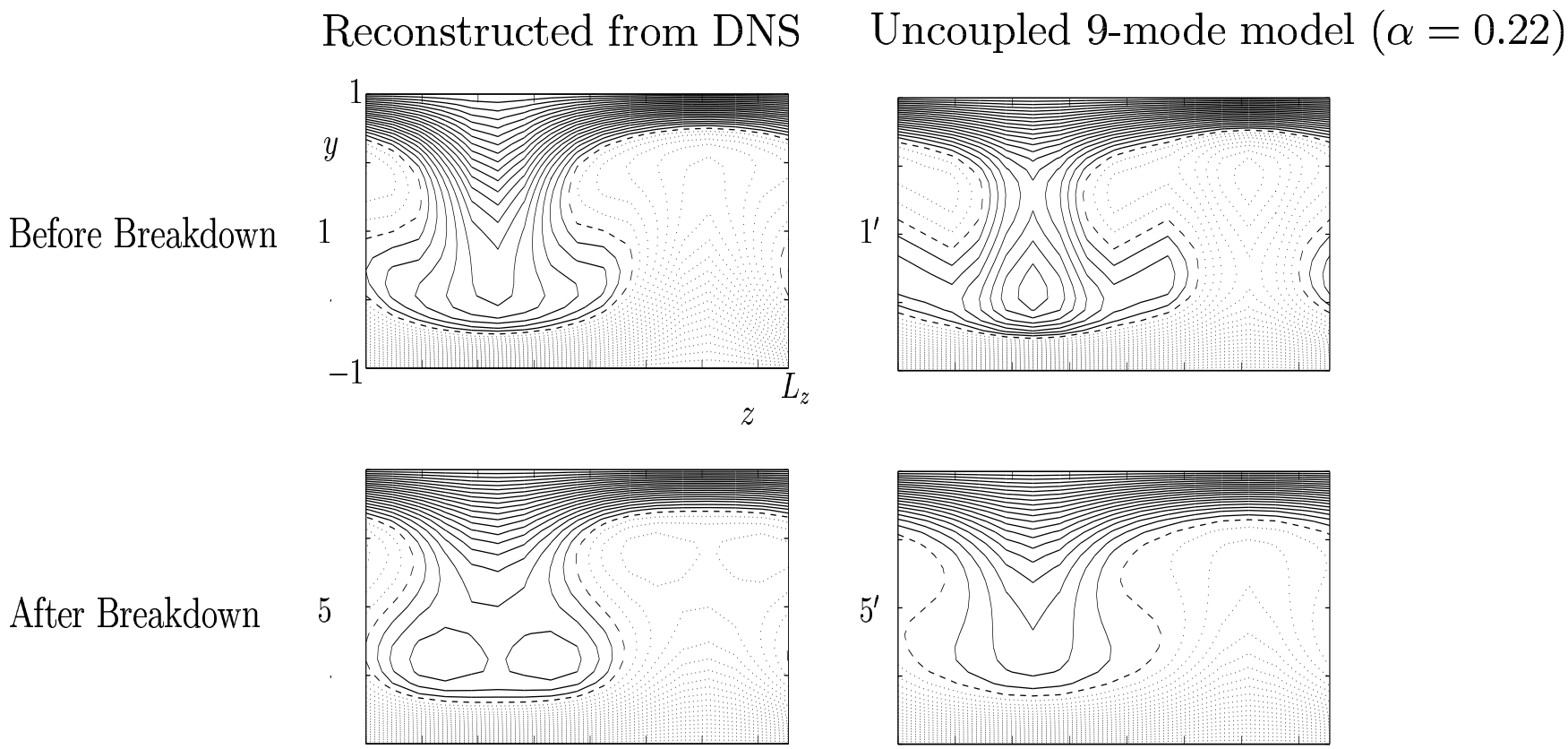
# Minimal flow unit: uncoupled models

- DNS vs. model streak-breakdown process



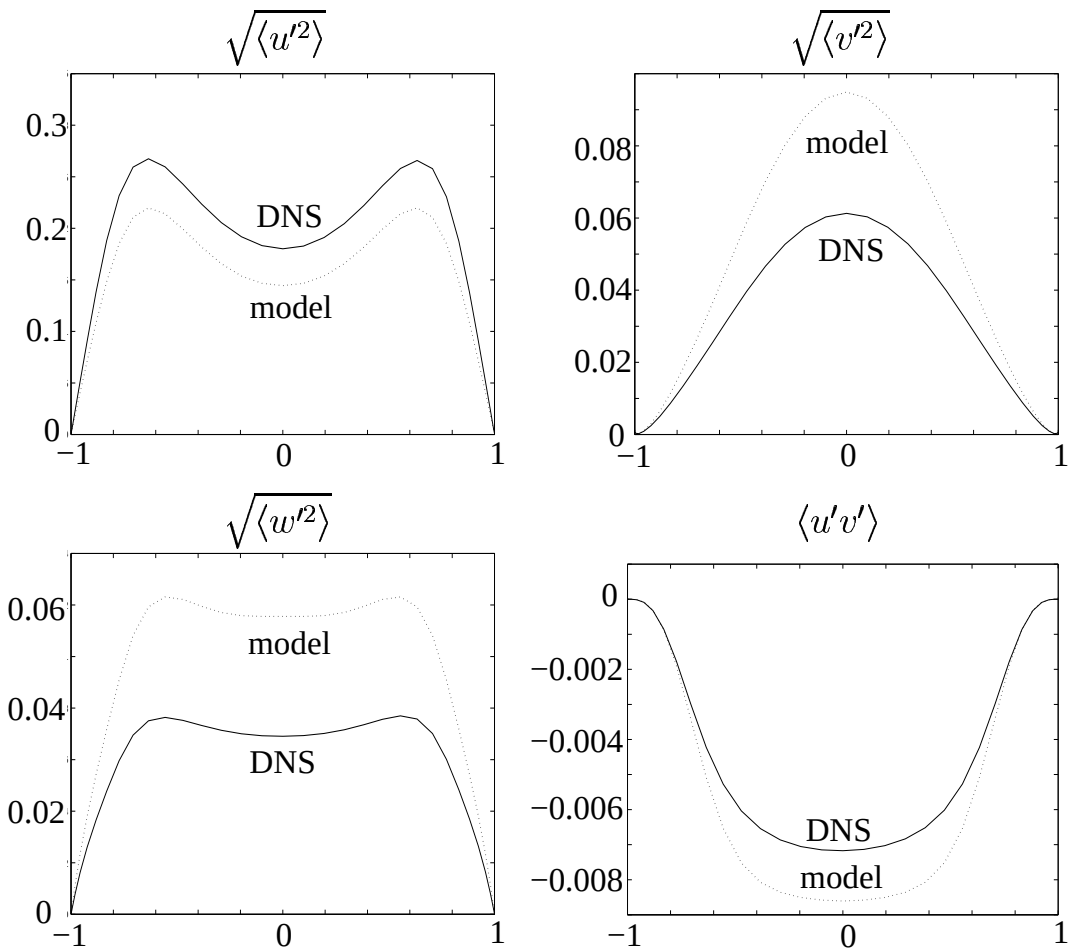
# Minimal flow unit: uncoupled models

- DNS vs. model streamwise velocity contours (including laminar state)



# Minimal flow unit: uncoupled models

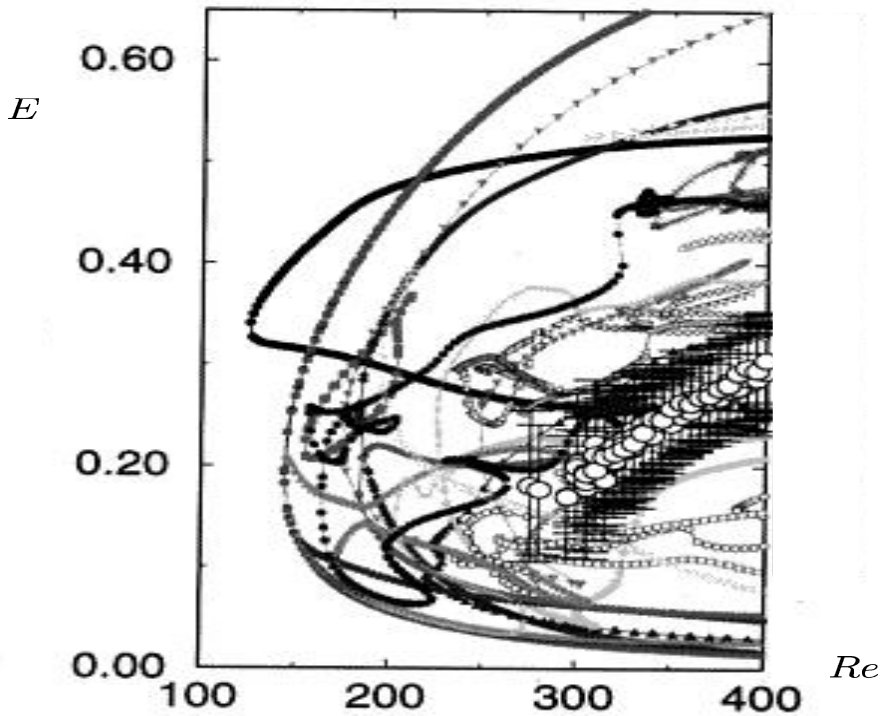
- DNS vs. 9-mode model turbulent statistics



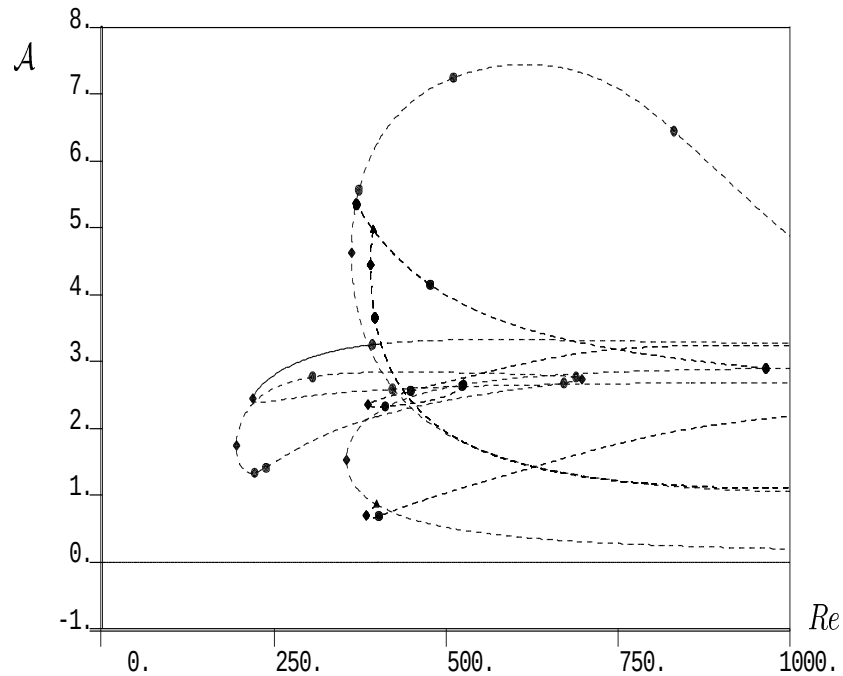
# Minimal flow unit: uncoupled models

- Branches of fixed points

From Schmiegel (PhD thesis)

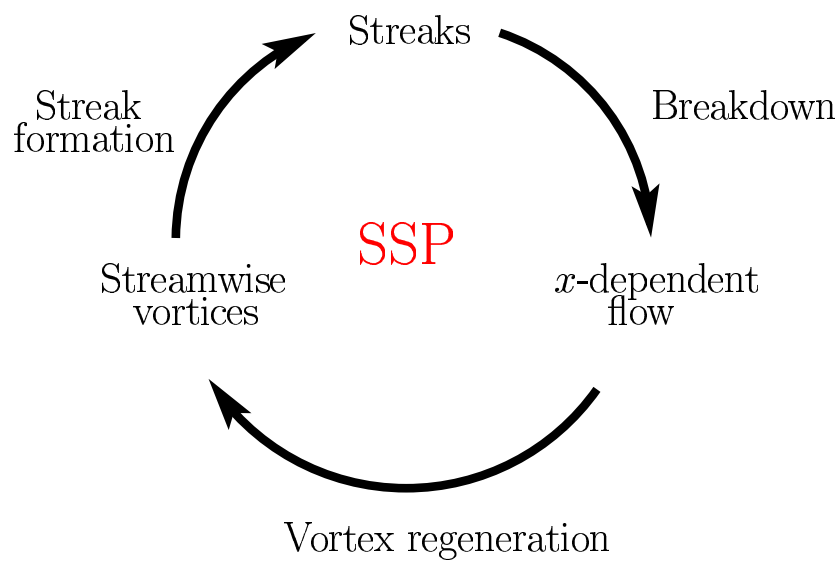



AUTO analysis of uncoupled 9-mode model



# Comparison with Waleffe (Phys. Fluid, 1995 & 1997)

- **was inspired by HKW to construct a low-dimensional model from physical arguments**




- **not PCF but "sinusoidal shear flow"**
- **predictions of 4-mode model**
  - correctly predicts stability of laminar state
  - certain coefficients: stable p.o. through *homoclinic* bifurcation.
  - velocity reconstructions from p.o. do not look like DNS
  - oefficients from Galerkin projection of NSE $\rightarrow$ *unstable* p.o.

# Conclusions

- **Have considered PCF in the MFU**
  - applied the POD - found almost all energy captured by first few modes
  - despite this, found that $\approx 1000$ modes required to reproduce DNS
- **Constructed two low-dimensional models**
  - 6-mode coupled model; qualitatively correct periodic orbit, but laminar state was *unstable*
  - 9-mode uncoupled model; periocdic orbit co-existing with stable laminar state
  - both models showed good quantative agreement with DNS time-scales, RMS modal velocities
  - good qualitative agreement with DNS streak-breakdown contours and statistics
  - Phase relationships of modal amplitudes better for coupled 6-mode model

# Possible improvements

- **including more modes**
  - e.g., more POD families, linear terms $\longrightarrow$ non-normal
  - can also solve the problem of the unstable laminar state
- **using symmetry ideas to get "better" POD modes**
  - **Rowley and Marsden (Physica D, 2000)**
- **better model for losses to neglected modes**