Lyapunov exponents, invariant manifolds and transport

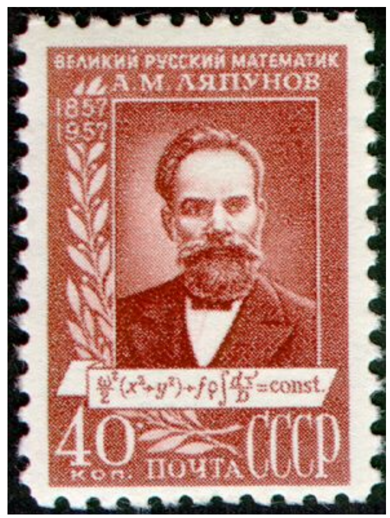
Kathrin Padberg

University of Paderborn

October 22, 2003

Motivation

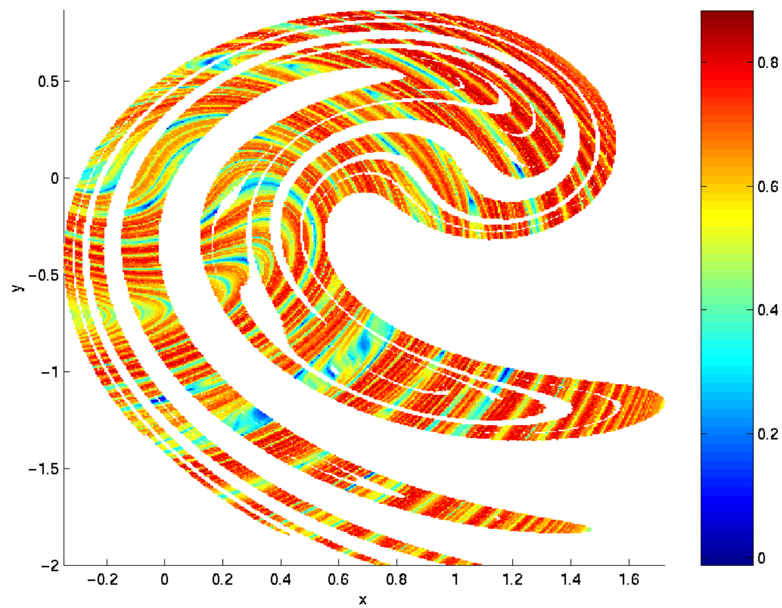
- dominant Lyapunov exponent is a measure of the chaoticity of an attractor
- global, asymptotic quantity - independent of the initial condition for μ-almost all initial points
- in practice: finite time Lyapunov exponents - depend on initial condition and the integration time



Russian stamp (1957) on the occasion of Lyapunov's 100th anniversary

Example: Ikeda attractor

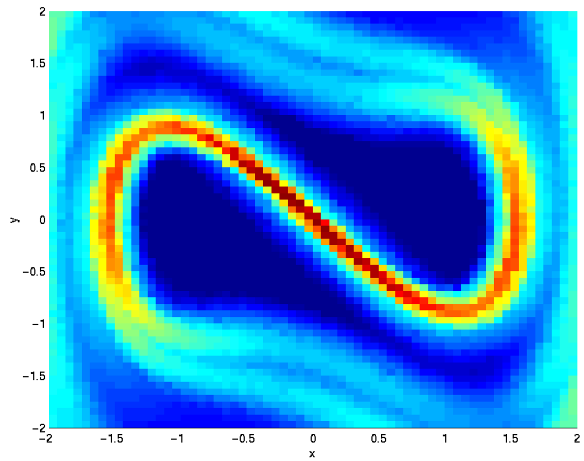
- distribution of finite time Lyapunov exponents (N=10 iterates)



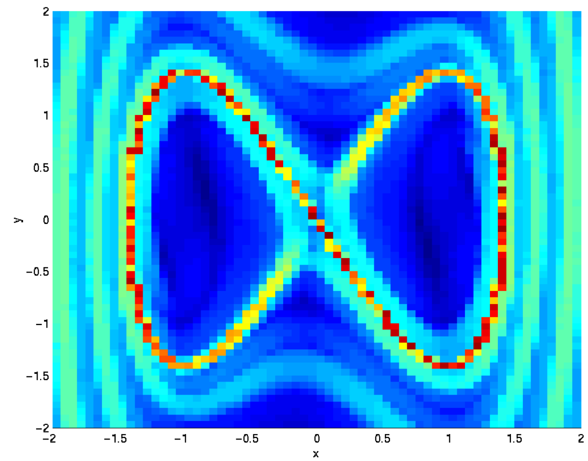
- some areas on the chaotic attractor are more predictable than others
- predictability analysis of this kind goes back to Lorenz (1965)

More examples

- distribution of finite time Lyapunov exponents (integration time T=5) for the Duffing system and a double well potential



Duffing



Double well potential

- detection of hyperbolic structures and their stable manifolds (e.g. Haller, 2001)

In this talk ...

- Basic ideas about Lyapunov exponents, local Lyapunov exponents and finite time Lyapunov exponents and their characteristics
- Why do finite time Lyapunov exponents pick up those particular structures?
- Many examples on how we can make use of this concept in a set oriented analysis of dynamical systems:
    - detection
    - extraction
- Expansion and graphs
- Future work and open questions

Evolution of a small perturbation

- Let g be a diffeomorphism on a compact manifold M of dimension l:

$$x_{k+1} = g(x_k),\ x_k \in M$$

- Consider a small (infinitesimal) perturbation ε_0 in the initial condition x_0. With $y_0 = x_0 + \varepsilon_0$ we obtain

$$\begin{aligned} y_1 &= x_1 + \varepsilon_1 \\ &= g(y_0) \\ &= g(x_0) + Dg(x_0) \cdot \varepsilon_0 + \text{ h.o.t.}, \end{aligned}$$

where $Dg(x_0)$ is the total derivative of g at x_0.

$$\Rightarrow \varepsilon_1 = Dg(x_0) \cdot \varepsilon_0.$$

- For $k \in \mathbb{N}$ it follows inductively

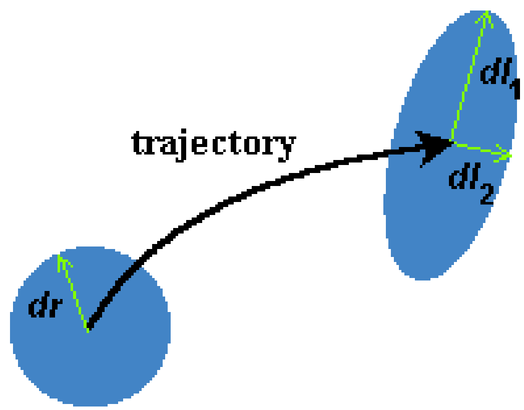
$$\begin{aligned}
\varepsilon_{k+1} &= Dg(x_k) \cdot \varepsilon_k \\
&= \prod_{i=0}^{k} Dg(x_i) \cdot \varepsilon_0 \\
&= Dg^k(x_0) \cdot \varepsilon_0 .
\end{aligned}$$

- So the evolution of a small displacement ε_0 is governed by the linearized dynamical system.

Lyapunov exponents

- Lyapunov exponents (LE) measure the convergence or divergence of infinitesimal perturbations in the initial condition

$$\lambda_i = \lim_{k \to \infty} \frac{1}{k} \log ||Dg^k(x) \cdot v_i||, \; v_i \in T_x M$$



- Their existence is guaranteed by the Multiplicative Ergodic Theorem.
- On an attractor LEs are independent of the initial condition x for μ-almost all x, where μ is an ergodic measure.

Dominant Lyapunov exponent

- In most applications, one is only interested in λ_1, the dominant Lyapunov exponent:

$$\begin{aligned}
\lambda_1 &= \lim_{N\to\infty} \frac{1}{N} \log ||Dg^N(x_0)|| \\
&= \lim_{N\to\infty} \frac{1}{N} \log || \prod_{n=0}^{N-1} Dg(x_n)|| \\
&= \lim_{N\to\infty} \frac{1}{2N} \log[\lambda_{max}((Dg^N(x_0))^\top Dg^N(x_0))],
\end{aligned}$$

  where $||\cdot||$ denotes the spectral norm.

- If $\lambda_1 > 0$ then the attractor is chaotic, if $\lambda_1 \leq 0$ we have regular dynamics (periodic or quasi-periodic attractor).

- Basically there are two different approaches to approximate the dominant LE: temporal (e.g. Dieci, van Vleck) or spatial averages (e.g. Froyland et al.; Aston, Dellnitz).

Local Lyapunov exponents

- From now on: Let g be a diffeomorphism on a compact manifold M.
- For every $x \in M$ the limit

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log ||Dg^n(x)||$$

  exists and is called the local Lyapunov exponent (LLE) w.r.t. x.
- In case of a chaotic attractor $\lambda(x) \neq \lambda_1$ applies only to a set of measure zero.
- $\lambda(x)$ is constant along orbits.

- Let $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{p-1}\}$ be a periodic orbit of period p. Then

$$\lambda(\bar{x}_i) = \frac{1}{p} \log \rho(Dg^p(\bar{x}_0)) \text{ for } i = 0, ..., p-1.$$

Here $\rho(A) = \max_\lambda\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

Proof: Let $p = 1$.

$$\begin{aligned}
\lambda(\bar{x}) &= \lim_{n\to\infty} \frac{1}{n} \log ||Dg^n(\bar{x})|| \\
&= \lim_{n\to\infty} \frac{1}{n} \log ||Dg(\bar{x})^n|| \\
&= \lim_{n\to\infty} \log ||Dg(\bar{x})^n||^{\frac{1}{n}} \\
&= \log \rho(Dg(\bar{x}))
\end{aligned}$$

(Use that $\lim_{n\to\infty} ||A^n||^{\frac{1}{n}} = \rho(A)$ for any consistent matrix norm.)

Now let $p \in \mathbb{N}$.

$$\begin{aligned}
\lambda(\bar{x}_0) &= \lim_{n\to\infty} \frac{1}{n} \log ||Dg^n(\bar{x}_0)|| \\
&= \lim_{n\to\infty} \frac{1}{n} \log || \prod_{i=0}^{n-1} Dg(\bar{x}_{i \,\mathsf{mod}_p})|| \\
&= \lim_{n\to\infty} \frac{1}{p \cdot n} \log || \prod_{i=0}^{n-1} \prod_{i=0}^{p-1} Dg(\bar{x}_i)|| \\
&= \lim_{n\to\infty} \frac{1}{p \cdot n} \log ||(Dg^p(\bar{x}_0))^n|| \\
&= \lim_{n\to\infty} \frac{1}{p} \log ||(Dg^p(\bar{x}_0))^n||^{\frac{1}{n}} \\
&= \frac{1}{p} \log \rho(Dg^p(\bar{x}_0))
\end{aligned}$$

Expansion rates (finite time Lyapunov exponents)

- In practice we deal with finite time Lyapunov exponents:
- The expansion rate is defined as

$$\Lambda(N, x_0) = \frac{1}{N} \log || \prod_{n=0}^{N-1} Dg(x_n) ||,$$

i.e. $\lim_{N\to\infty} \Lambda(N, x_0) = \lambda(x_0)$.

- Let $N \in \mathbb{N}$ be fixed. Then $\Lambda(N, x)$ depends continuously on x.
- Let $\omega(x) = \{\bar{x}_0, \bar{x}_1, ..., \bar{x}_{p-1}\}$ be a periodic orbit of period p. Then

$$\lambda(x) = \lambda(\bar{x}_0) = \frac{1}{p} \log \rho(Dg^p(\bar{x}_0)).$$

Direct expansion rate

- Consider finite instead of infinitesimal perturbations:
- The direct expansion rate is given by

$$\Lambda_\varepsilon(N, x_0) := \frac{1}{N}\log\left(\max_{\{x:||x_0-x||=\varepsilon\}} \frac{||g^N(x_0) - g^N(x)||}{\varepsilon}\right).$$

- Here

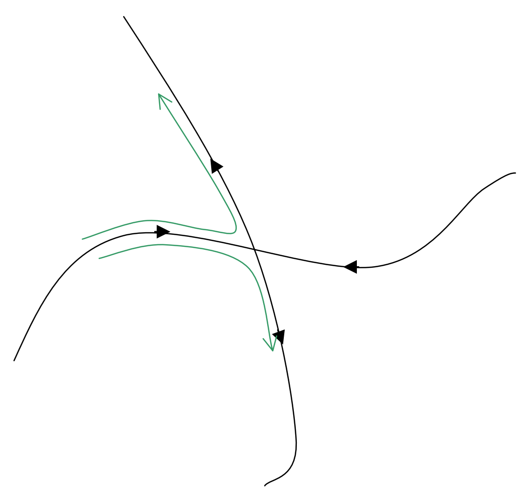
$$\max_{\{x:||x_0-x||=\varepsilon\}} \frac{||g^N(x_0) - g^N(x)||}{\varepsilon}$$

  is the maximal relative dispersion of a particle pair (e.g. Provenzale, Bowman).
- The expansion and direct expansion rates are related in the following way:

$$\lim_{\varepsilon\to 0} \Lambda_\varepsilon(N, x_0) = \Lambda(N, x_0).$$

Summary of theory

- LLEs are constant along orbits and determined by the limit set of the orbit.
- The expansion rate $\Lambda(N, \cdot\,)$ converges pointwise to $\lambda(\cdot\,)$ for $N \to \infty$.
- $\Lambda(N, x)$ depends continuously on x for fixed N.
- Thus, for large enough $N > 0$, local maxima in the scalar field $\Lambda(N, x)$ may be an indicator for dominant hyperbolic periodic points and their stable manifold.



- Proposition: Let $A \subset M$ be a compact forward invariant set. Let $\lambda(\bar{x}) = \max_{x \in A} \lambda(x)$ for a hyperbolic fixed point $\bar{x} = g(\bar{x})$. Consider some neighborhood $B_r(\bar{x}) \subset A$ of $\bar{x}$. Then for $\varepsilon > 0$ there is $N > 0$ and $(B_r(\bar{x}) \cap W^s(\bar{x})) \subset U \subset B_r(\bar{x})$ such that $|\Lambda(N, x) - \lambda(\bar{x})| < \varepsilon$ for all $x \in U$ and $\Lambda(N, x) > \Lambda(N, y)$ for all $x \in U$ and $y \notin U$.

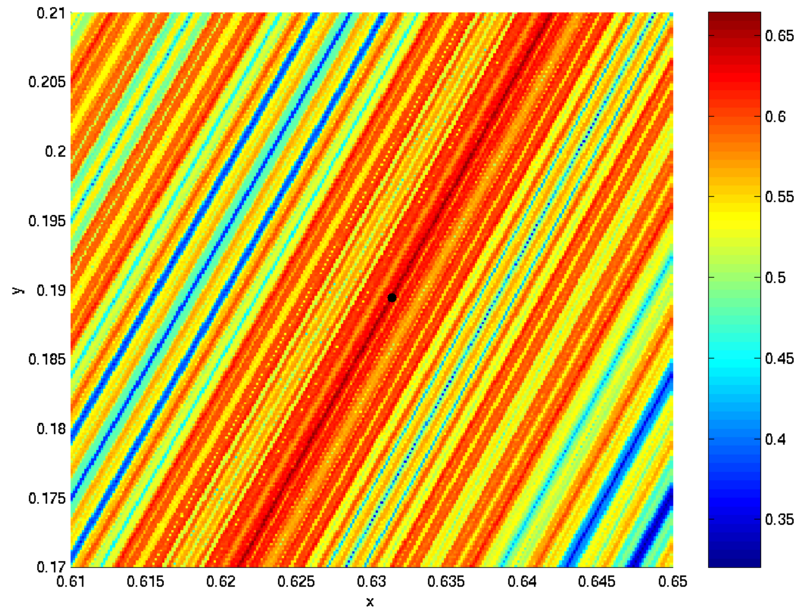


Figure 1: Expansion rates (N=20) in a neighborhood of the saddle point for the Hénon map.

Numerical approximation of expansion rates

- Given: box collection $\mathcal{B}_k$ that covers of our region of interest (e.g. attractor, chain recurrent set, compact set in phase space).

- Expansion rate for a box B:

$$\delta(N, B) := \max_{x_0 \in B} \Lambda(N, x_0).$$

- Direct expansion rate for B:

$$\delta_\varepsilon(N, B) := \max_{x_0 \in B} \Lambda_\varepsilon(N, x_0).$$

- $\delta(N, B) \to \Lambda(N, x)$ and $\delta_\varepsilon(N, B) \to \Lambda_\varepsilon(N, x)$ for $\mathrm{diam}(B) \to 0$ and $B \to x$.

- In practice we compute an approximation of $\delta(N, B)$ or $\delta_\varepsilon(N, B)$ using test point strategies.

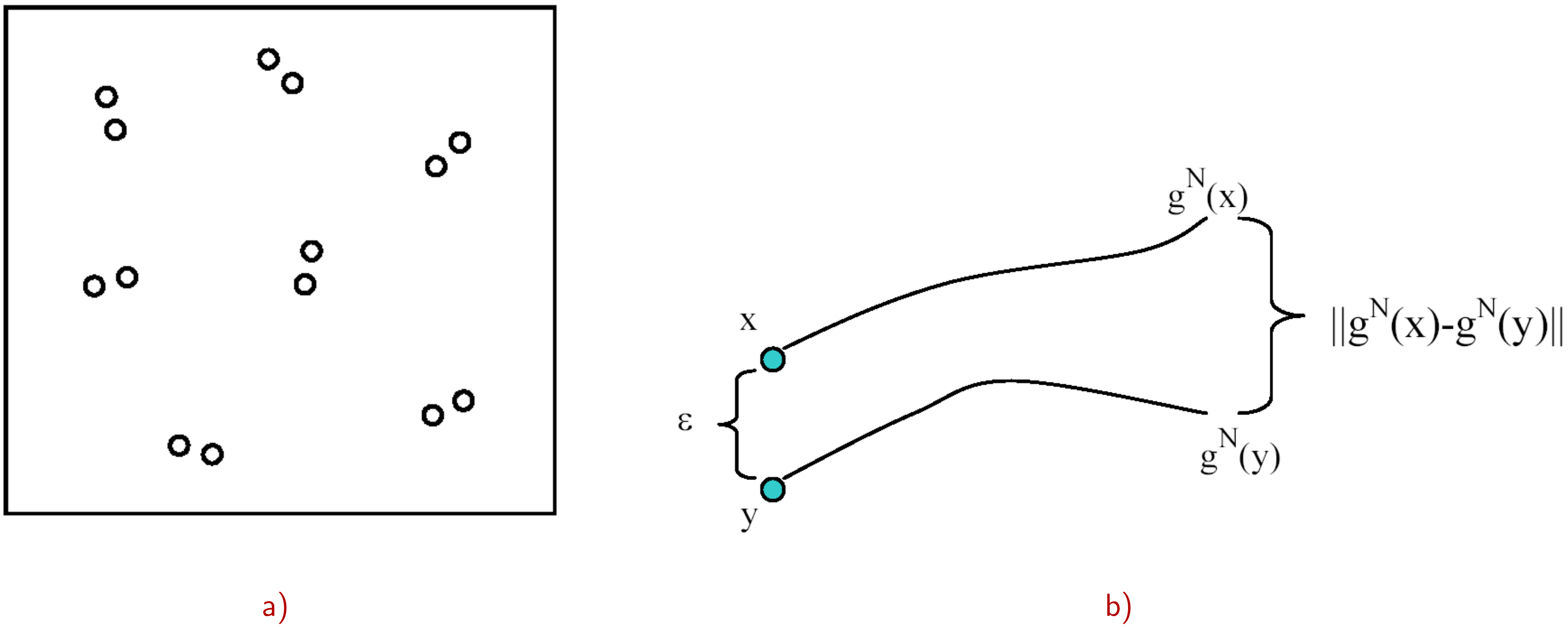


Figure 2: a) Choice of test points pairs in a box. b) Computation of the relative dispersion of a pair of initial conditions.

Hénon map

$$h(x, y) = (1 + y - 1.4x^2, 0.3x)$$

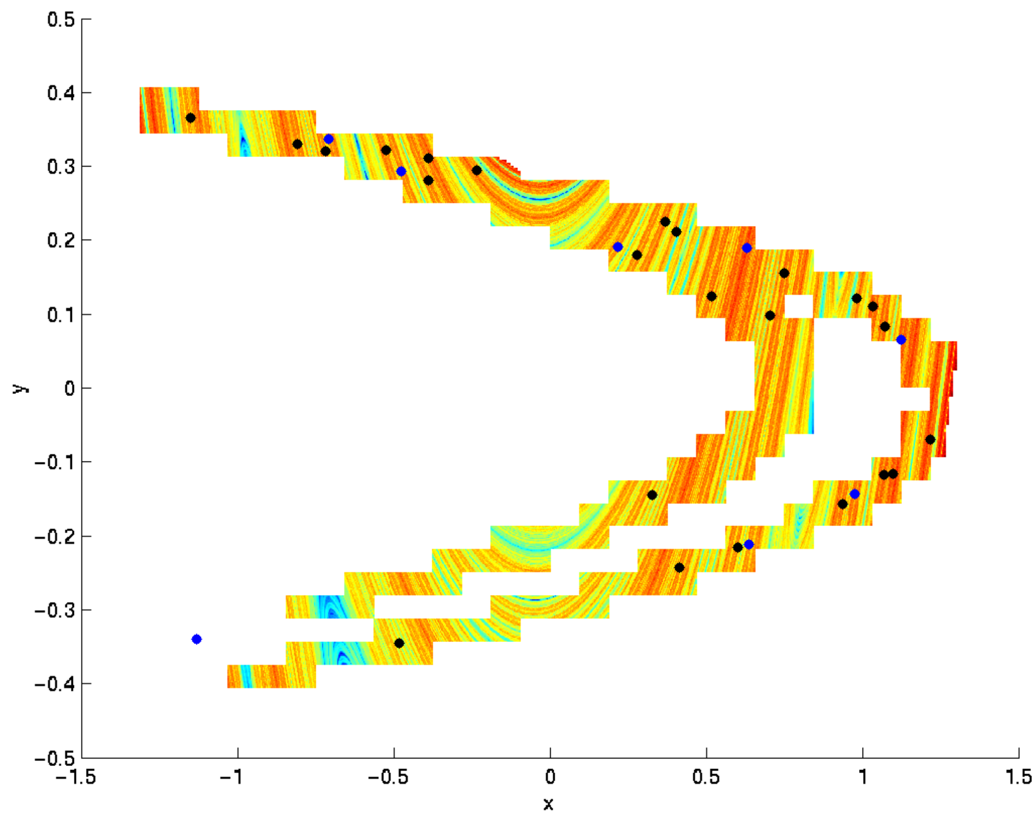
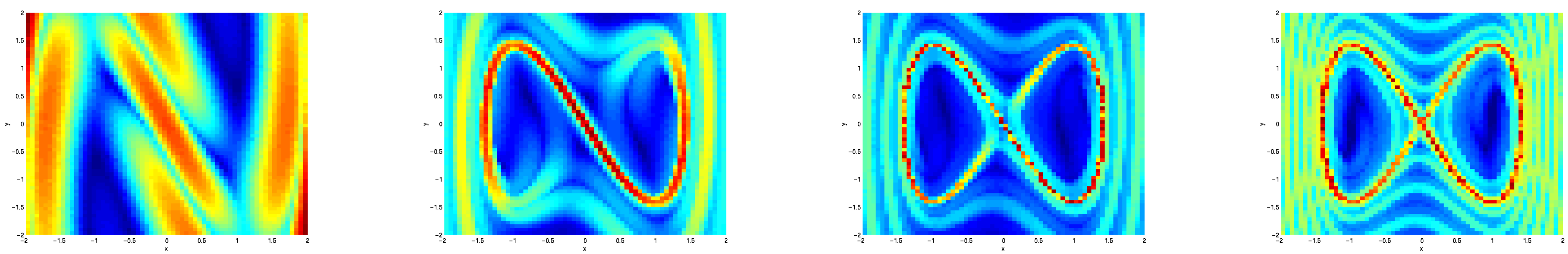


Figure 3: Direct expansion rate (N=20) applied to part of the domain of the attractor.

Double-well potential

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -4x(x^2 - 1) \end{aligned}$$

This Hamiltonian system has three fixed points: two centers at $(\pm 1, 0)$ and a saddle at $(0, 0)$. Two homoclinic orbits connect the saddle with itself. The centers are surrounded by stable periodic orbits and so is the union of the two saddle loops.



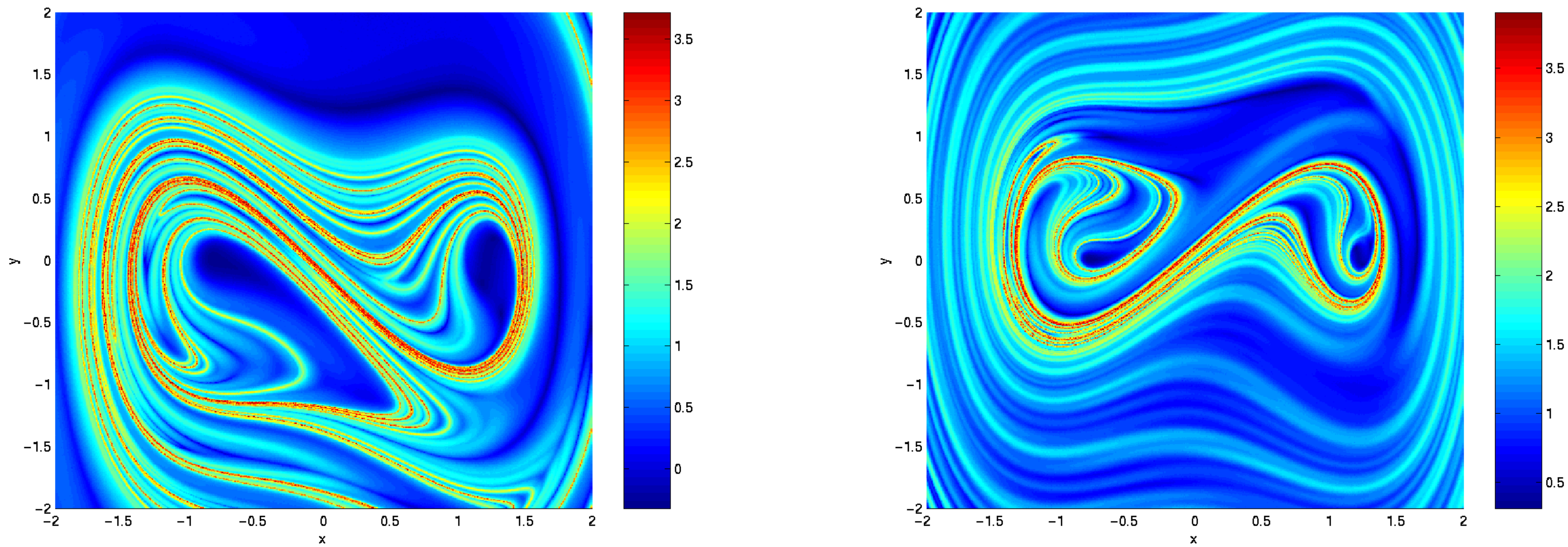
a) b) c) d)

Figure 4: Direct expansion rates for the double-well potential. a) T=1; b) T=3; c) T=5; d) T=10.

Duffing oscillator

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos \omega t.$$

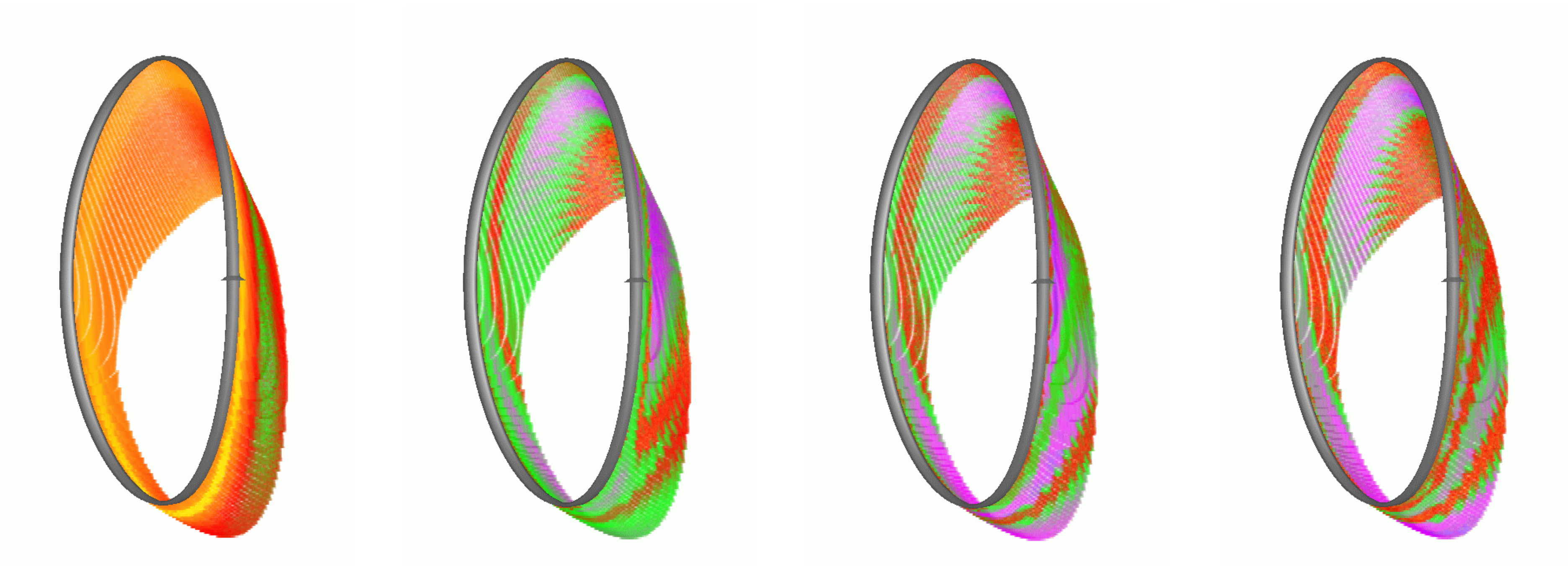
We choose $\gamma = 0.2$, $\delta = 0.1$ and $\omega = 1$ and consider the time-2π map.



a) b)

Figure 5: Direct expansion rates for the Duffing oscillator for N=3. a) Forward time, b) backward time iteration.

- We consider part of the unstable manifold of a Halo orbit about L_1 in the Sun-Earth system.



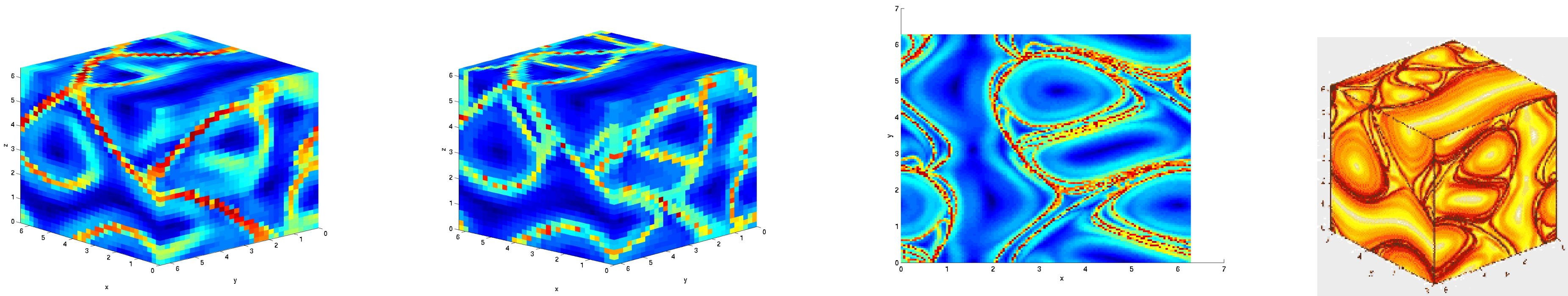
a) b) c) d)

Figure 6: Direct expansion rates for a box covering of part of the unstable manifold of a Halo orbit about L_1. a) T=1; b) T=3; c) T=5; d) T=10.

Steady ABC flow

$$\begin{aligned}\dot{x} &= A\sin z + C\cos y\\ \dot{y} &= B\sin x + A\cos z\\ \dot{z} &= C\sin y + B\cos x\end{aligned}$$

This class of flows is known as ABC (Arnold-Beltrami-Childress) flows. In our numerical studies we consider the cube $[0, 2\pi]^3$ and fixed the parameter values $A = \sqrt{3}$, $B = \sqrt{2}$, $C = 1$ (cf. Haller, 2001).

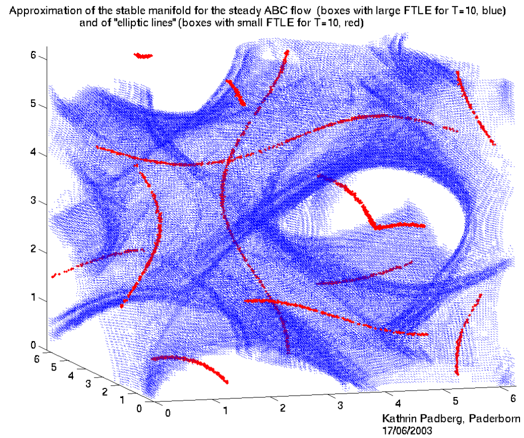


a) b) c) d)

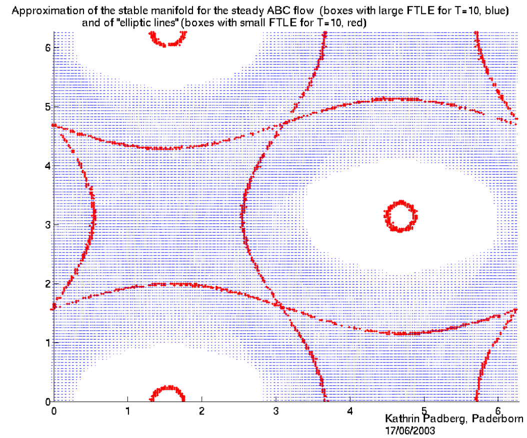
Figure 7: Direct expansion rates for the ABC-flow. a) T=3; b) T=5; c) T=10 ($y - z$ plane); d) computation by G. Haller (2001)

Steady ABC flow (cont.)

- Subdivision algorithm for the extraction of hyperbolic and elliptic structures.
- Selection criterion is the expansion rate of a box.



a)



b)

Figure 8: a) Regions in the ABC flow with high (blue) and zero (red) expansion, corresponding to hyperbolic and elliptic material lines. A multilevel approach is used to restrict the box covering to the relevant objects. b) Projection.

Duffing oscillator

- Subdivision algorithm to extract the stable manifold of a hyperbolic fixed point.

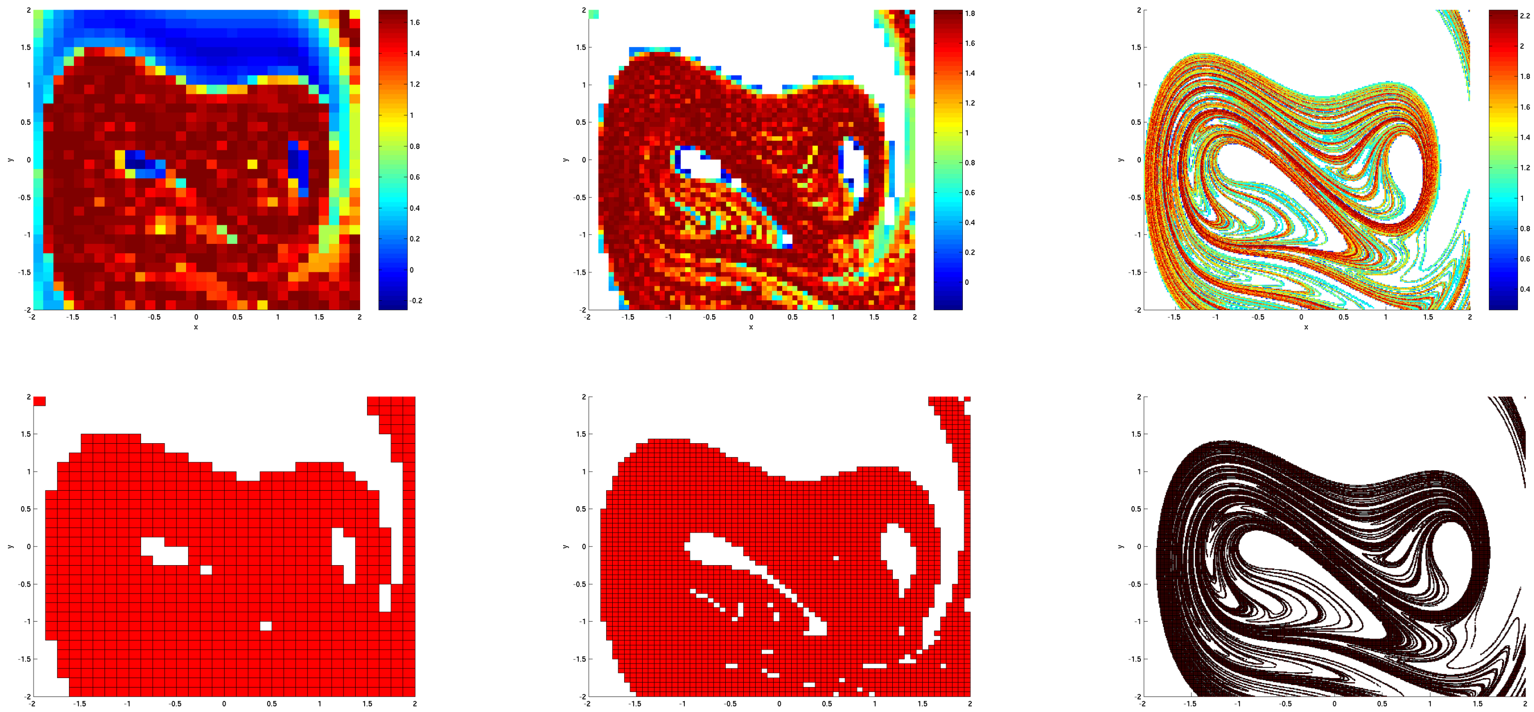
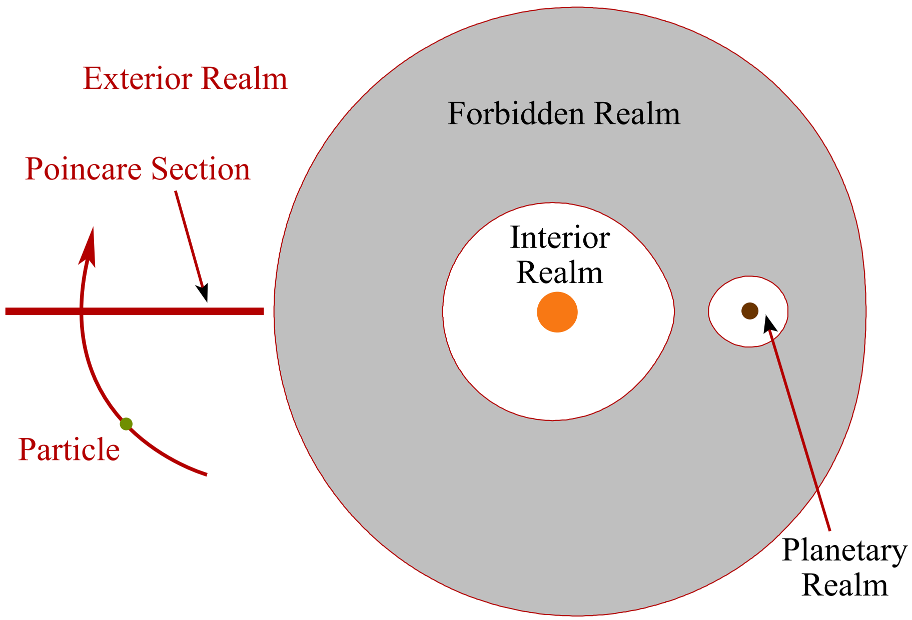


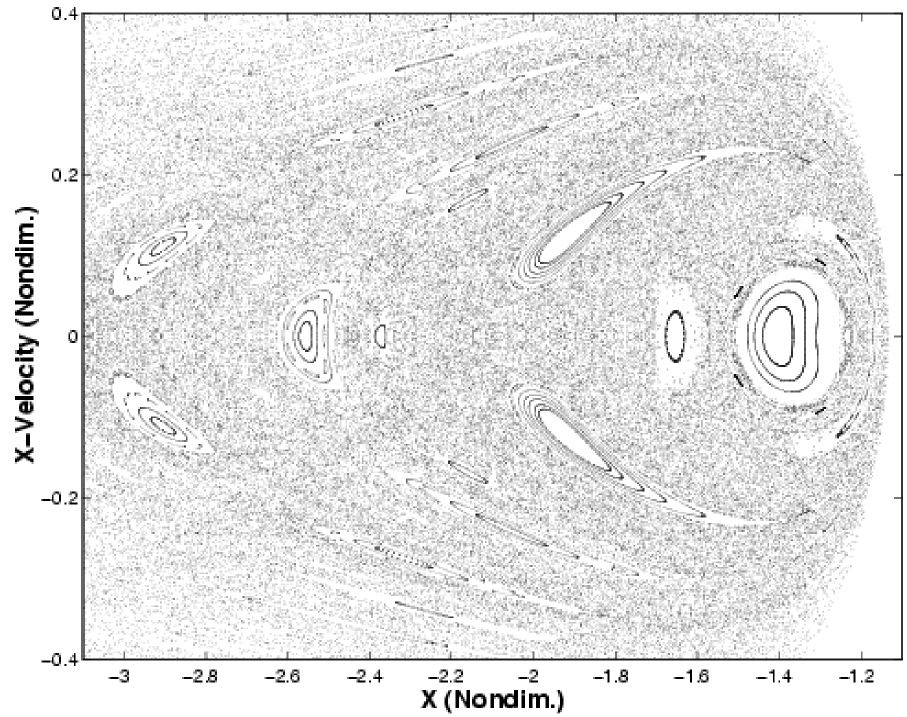
Figure 9: Illustration of a multilevel subdivision technique for the Duffing system (N=5).

Transport problem

- Consider Poincaré section in the Planar Circular Restricted Three Body Problem for the Sun-Jupiter-particle system, $C = 3.05$.



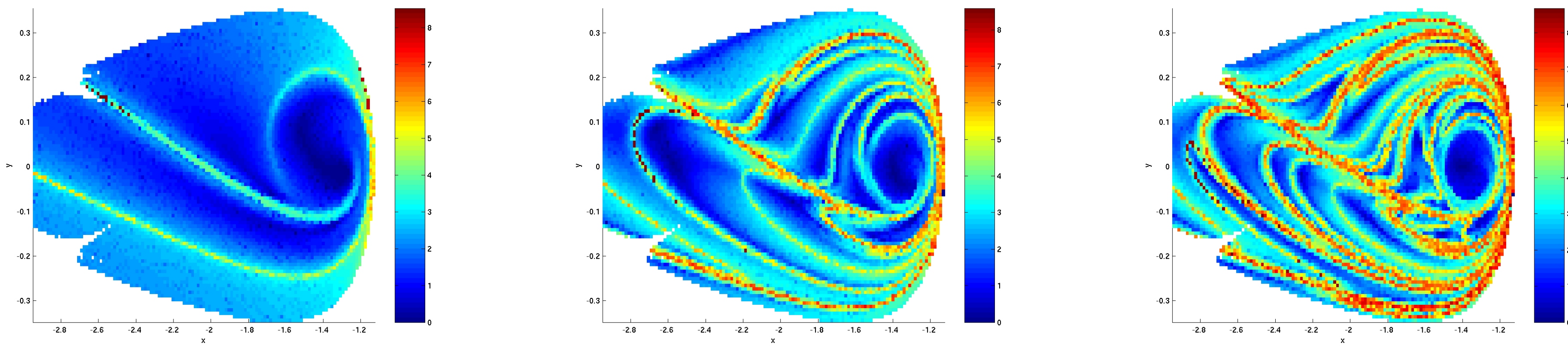
(a)



(b)

Figure 10: A Poincaré section of the flow in the restricted three-body problem. (a) The location of the Poincaré section. (b) The mixed phase space structure of the PCR3BP.

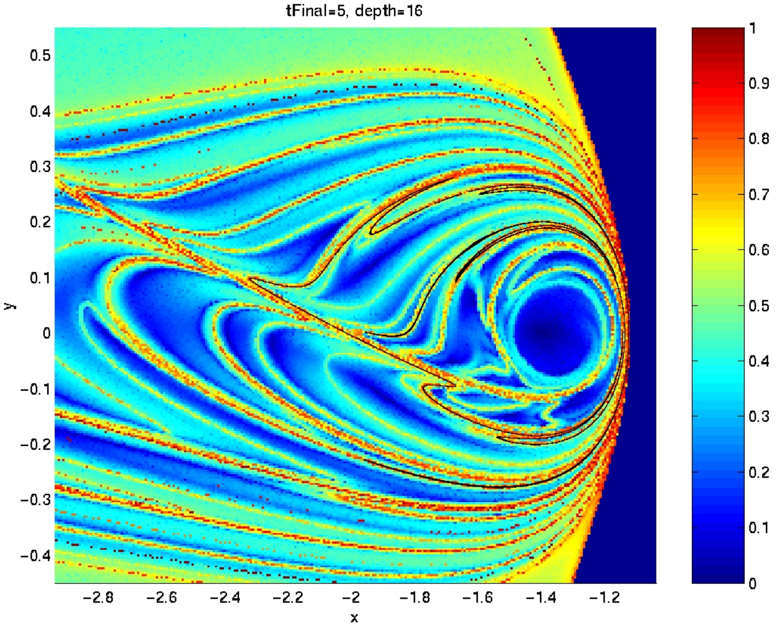
- Analysis of a box covering of the region of interest in the restricted three body problem.



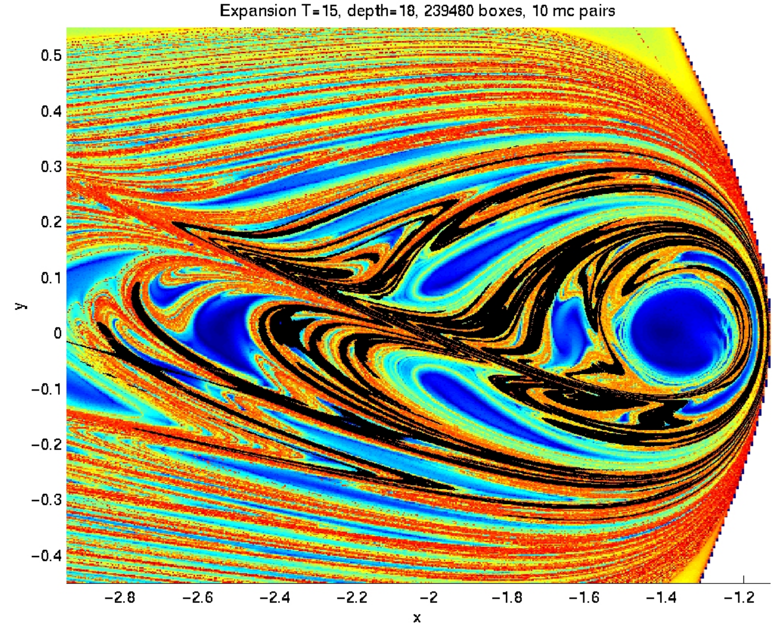
a) b) c)

Figure 11: Direct expansion rates for the transport problem. a) N=1; b) N=3; c) N=5.

Expansion rates and invariant manifolds



a)



b)

Figure 12: Direct expansion rates for the transport problem and part of the stable manifold of the saddle point. a) N=5; b) N=10.

- comparison of lobe dynamics and graph algorithms (PB-Caltech-JPL)

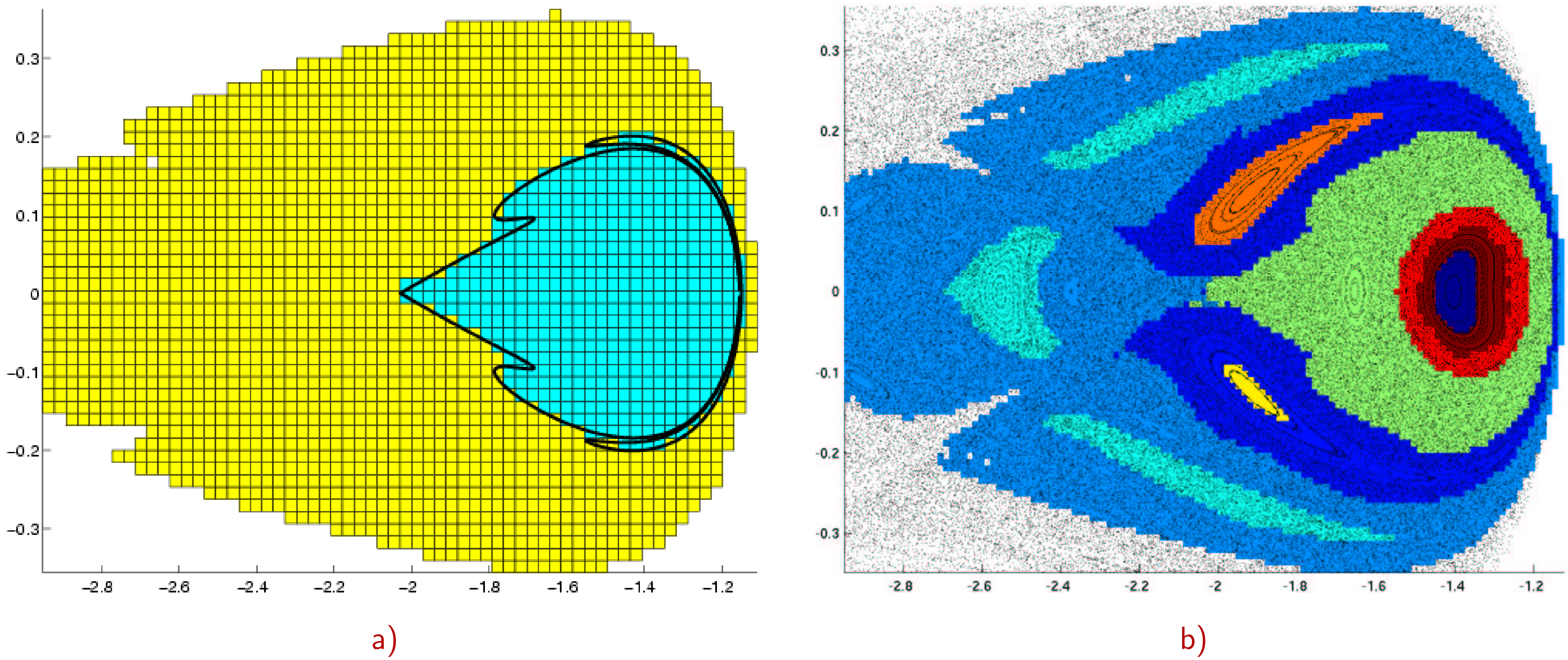
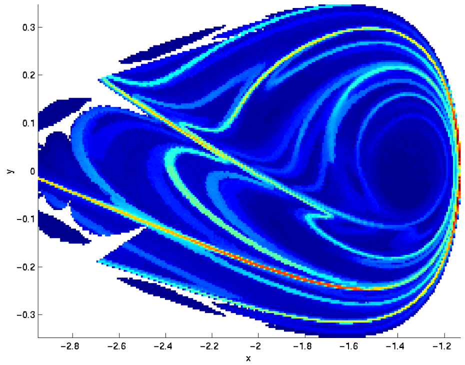


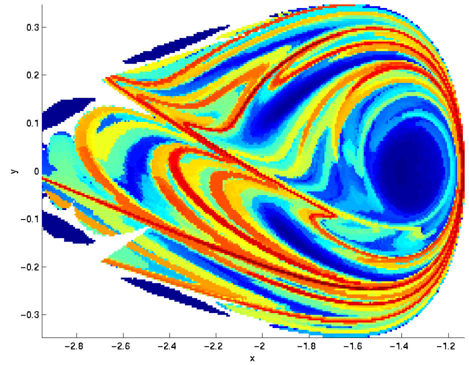
Figure 13: Transport example. a) Invariant manifolds of a hyperbolic fixed point and two-partitioning of the underlying graph of the discretized dynamical system. b) Mixed phase space structure and a partition into nine almost invariant sets.

- Heuristics to analyse the graph w.r.t. expansion (work with R. Preis): nodes that induce 'large' subgraphs U of a given diameter
  - number of nodes of U
  - weight of U
  - weight of internal edges
  - weight of external edges

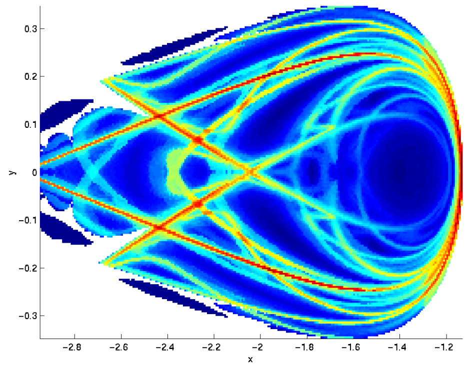
Transport example



a)



b)



c)

Figure 14: Expansion in a graph (transport problem) using the number of nodes in subgraphs of a specific diameter. a) Diameter 5; b) diameter 10; c) diameter 5 (undirected graph).

Open questions and future work

- optimal choice of N
- explanation of the dominance of period-1 points
- hierarchy of transport barriers
- proof of convergence for this particular subdivision algorithm
- application to real data
- graph theoretic analogue to saddle points and invariant manifolds