Collisions, Chaos and Periodic Orbits in the Anisotropic Manev Problem

Manuele Santoprete

Department of Mathematics
University of California Irvine

California Institute of Technology

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Preliminaries

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The Manev Model

- The Manev Model is a two body problem given by the potential energy

$$U = -\frac{A}{r} - \frac{B}{r^2},$$

where r is the distance between the particles and A, B are positive constants.

- This model describs the precession of the perihelion of Mercury with the same accuracy as general relativity.
- The Manev model also describes the relativistic dynamics of the Hydrogen atom, from a "classical" point of view.

The Anisotropic Problems

- The anisotropic problems , that replace the "flat" space with an anisotropic one, have been defined by Gutzwiller in the 1970s.
- Gutzwiller wanted to find an approximation of the quantum mechanical energy levels for a chaotic systems.
- He chose to study the Anisotropic Kepler Problem because
  - **It is chaotic**
  - **It is suitable to model physical phenomena encountered in semiconductors**

Anisotropic Manev Problem

- **The Anisotropic Manev Problem** (for example Craig et al. [1999]) describes the motion of two point masses in an anisotropic space under the influence of the Manev law.
- It was first proposed in the mid 1990s by F. Diacu to have a deeper understanding of the connections between classical, relativistic and quantum mechanics.

Physical Motivations

- Finding connections between Classical, Quantum and Relativistic Mechanics
- Describing a relativistic version of the Anisotropic Kepler Problem
- Describing gravitational models with anisotropic gravitational constants.
- Can be used as a toy model for anisotropic problems in physics, astronomy, celestial mechanics,...

Mathematical Motivations

- Studying its peculiar collision manifolds and its collision orbits
- Studying the mechanism responsible for the appearance of chaos
- Studying its discrete group of symmetry and the consequences of its existence.

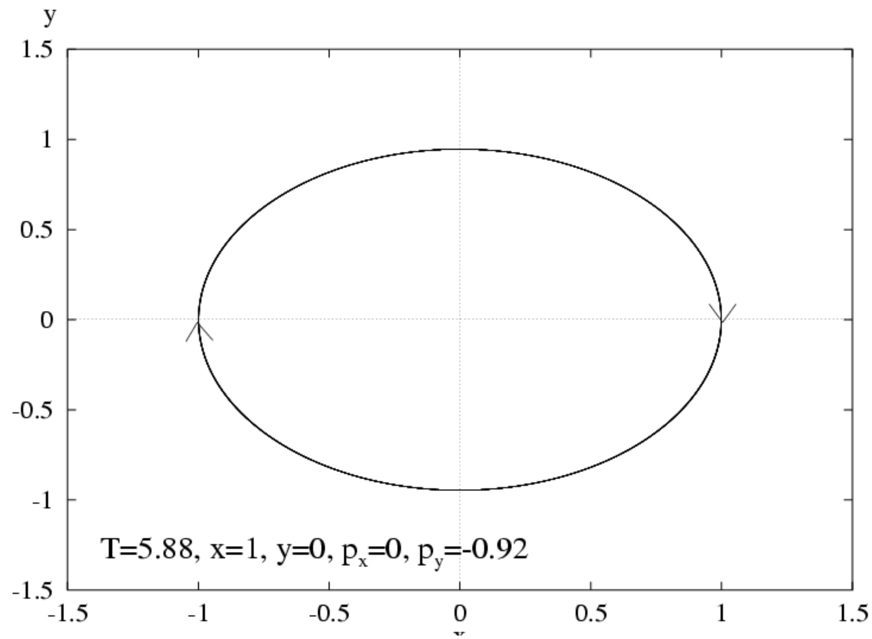
- The **Hamiltonian** is:

$$H_0 = \frac{1}{2}\mathbf{p}^2 - \frac{1}{\sqrt{\mu x^2 + y^2}} - \frac{b}{\mu x^2 + y^2}$$

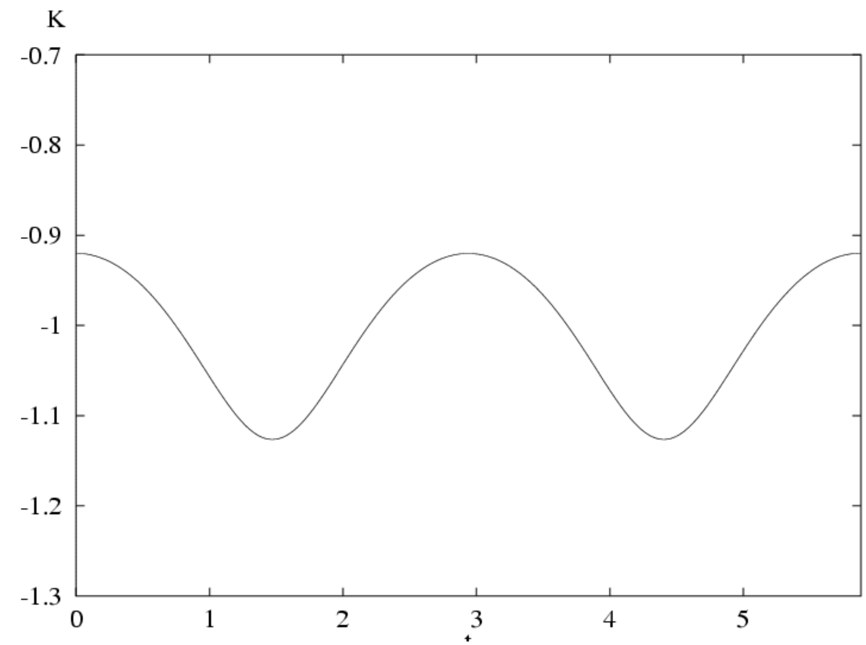
  where $b > 0$ and $\mu > 1$ are constant.

- The Hamiltonian is a first integral, but the angular momentum is not.

- This makes the dynamics of the problem interesting...and complicated!!



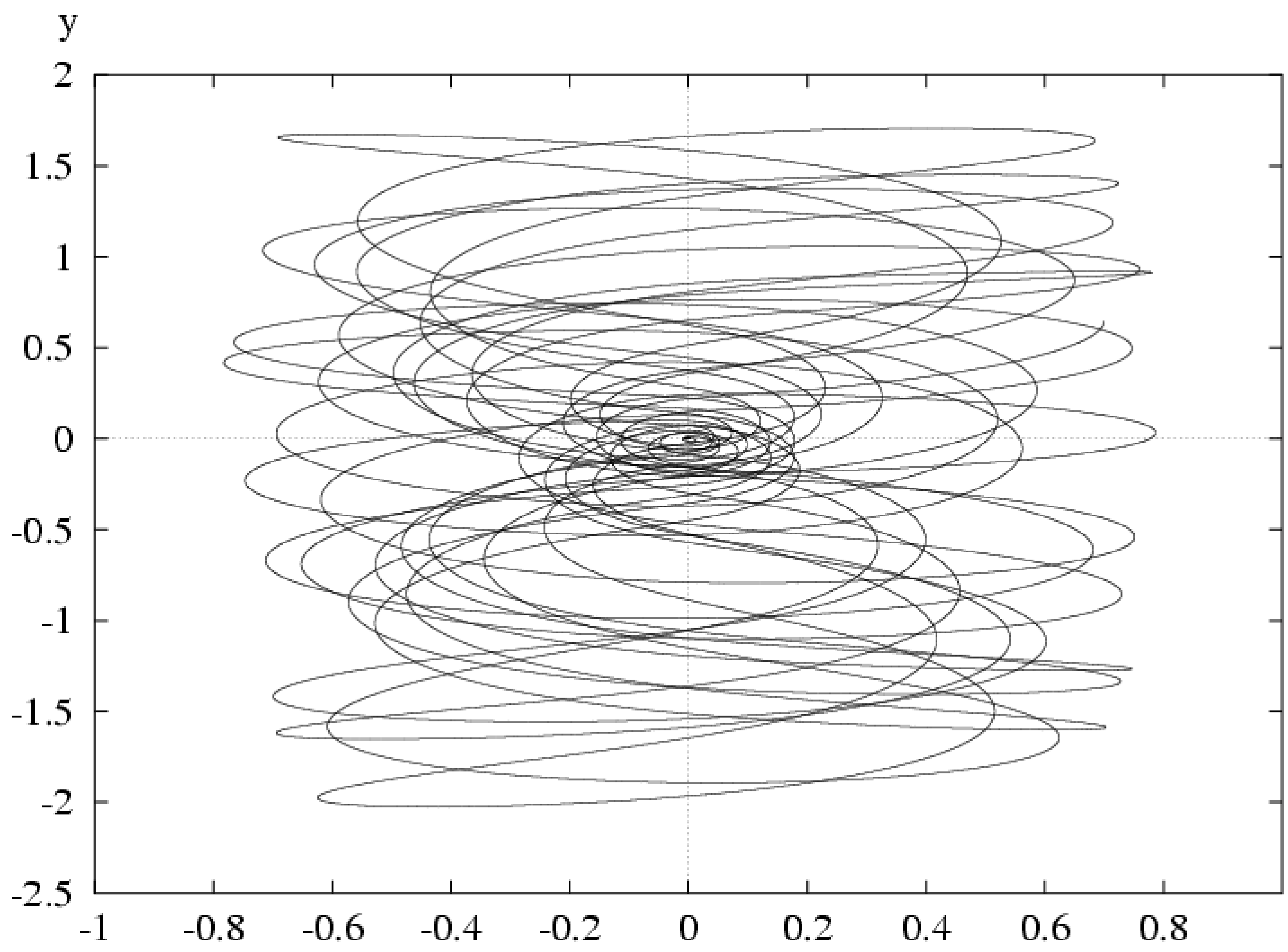
(a)



(b)

(a) A periodic orbit of the anisotropic Manev problem with $\mu = 1.5$, $b = 0.1$.

(b) The angular momentum K of the periodic orbit as a function of time.



A "chaotic" orbit that behaves erratically, obtained numerically for $\mu = 15$

Collision orbits and Chaos

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Singularities

- Solutions leading to collisions as well as those coming close to collisions are of particular interest because the qualitative structure of the phace space depends on their behavior.
- In the study of collision and near collision solutions it is helpful to transform the system using a method developed by McGehee.
- The idea is to "blow-up" the collision singularity, replace it with a so-called collision manifold and extend the phase space to it.

McGehee Coordinates

Consider the coordinate transformations

$$\begin{cases} r = |\mathbf{q}| \\ \theta = \arctan(y/x) \\ v = \dot{r}r = (xp_x + yp_y) \\ u = r^2\dot{\theta} = (xp_y - yp_x) \end{cases}$$

and the rescaling of time $d\tau = r^{-2}dt$. The new equation of motion are:

$$\begin{cases} r' = rv \\ v' = 2r^2h + r\Delta^{-1/2} \\ \theta' = u \\ u' = (1/2)(\mu - 1)(r\Delta^{-3/2} + 2b\Delta^{-2}) \sin 2\theta \end{cases}$$

and the energy relation takes the form

$$u^2 + v^2 - 2r\Delta^{-1/2} - 2b\Delta^{-1} = 2r^2h,$$

where $\Delta = \mu \cos^2\theta + \sin^2\theta$. The prime denotes differentiation with respect to τ.

The Collision Manifold

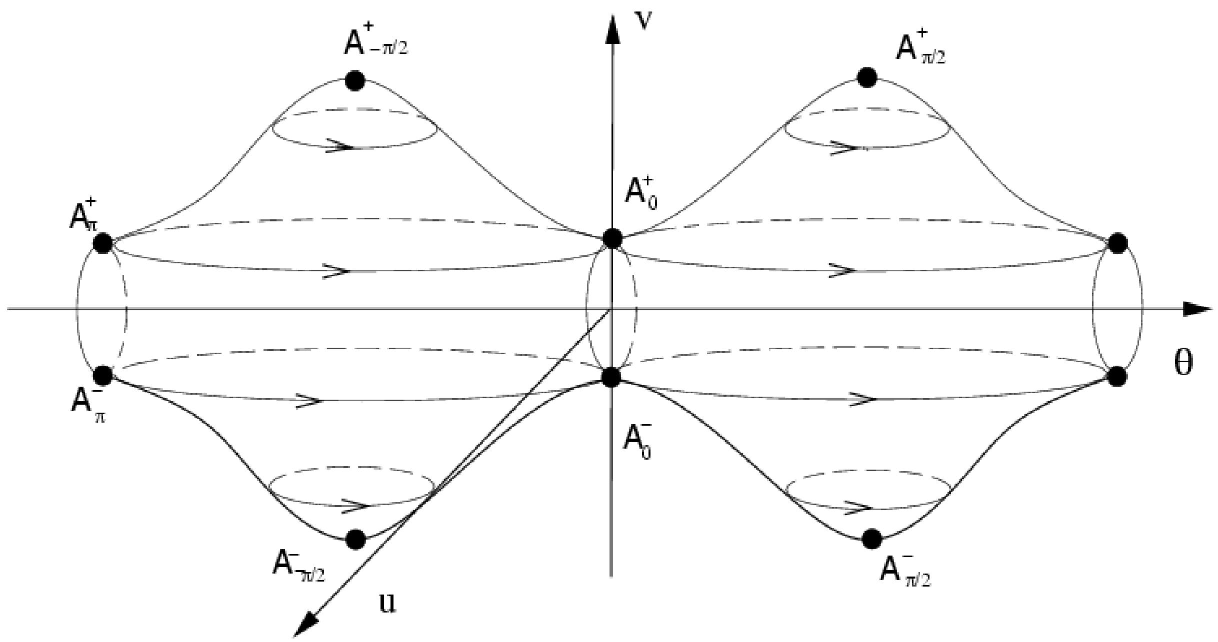
- The set

$$C = \{(r, v, \theta, u) | r = 0 \text{ and the energy relation holds}\}$$

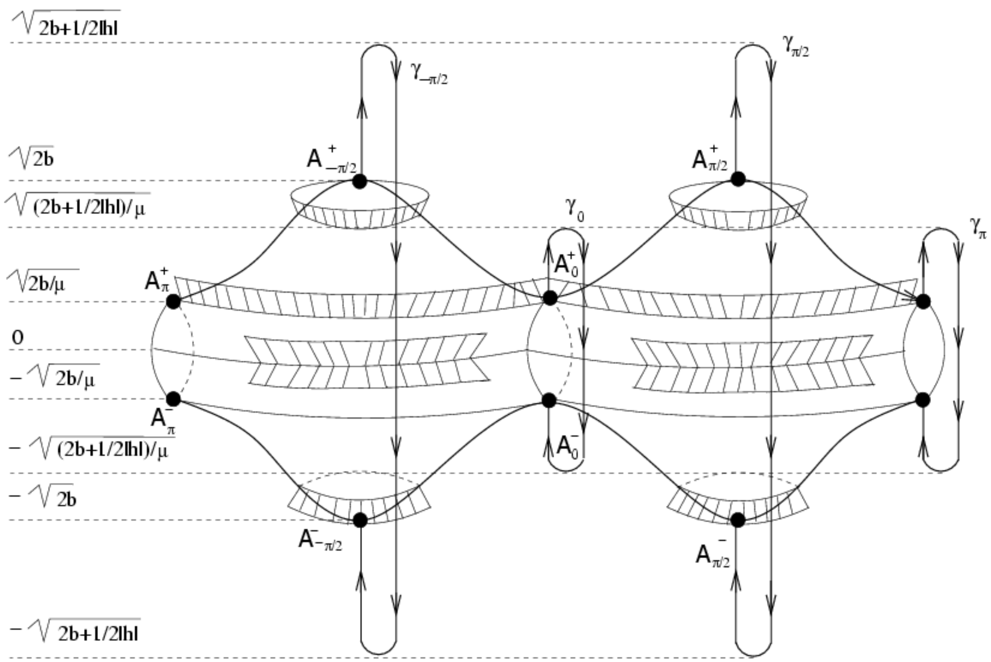
is the collision manifold.

- This 2-dimensional manifold, embedded in $\mathbb{R}^3 \times S^1$, is homeomorphic to a torus and it is given by the equations

$$r = 0 \quad \text{and} \quad u^2 + v^2 = 2b\Delta^{-1}.$$



- On the collision Manifold
  - **The solutions lie on the level curves $v =$ constant of the torus C.**
  - **There are eight equilibria and eight heteroclinic orbits.**
  - **All the other solutions are periodic.**

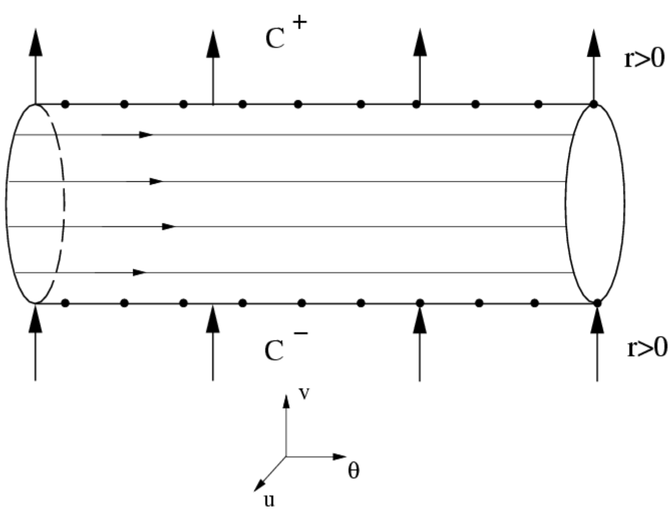


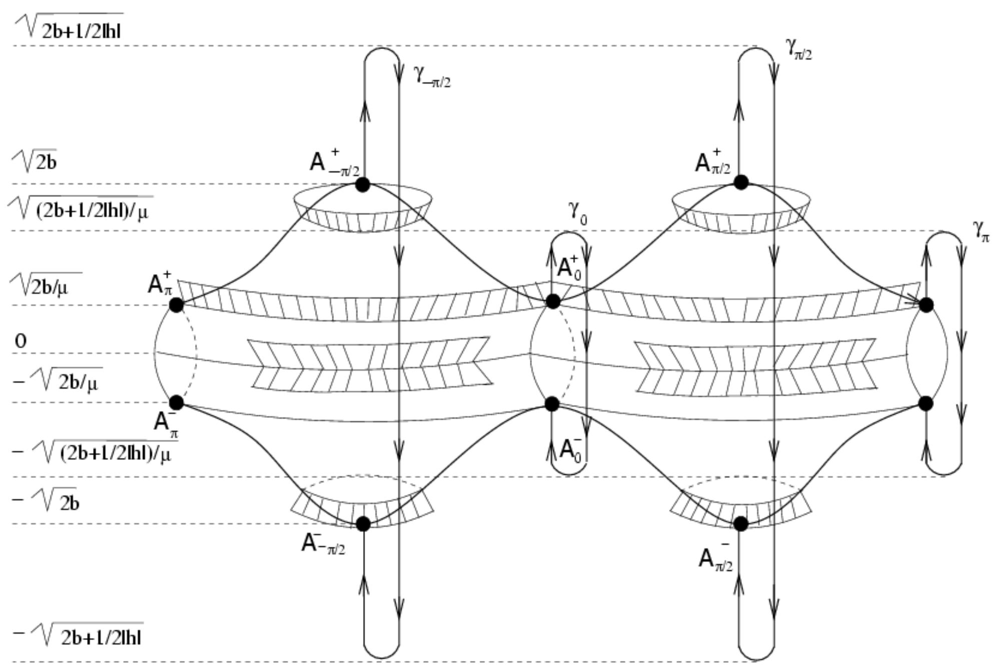
- Outside the collision manifold:
  - Each periodic orbit (except on the equator) has either a two dimensional stable or unstable manifold.
  - The periodic orbits on the equator have a positively and a negatively asymptotic set.

Heteroclinic Orbits

- When $\mu > 1$ most of the heteroclinic orbits of the Manev problem are destroyed. Some however persist for all μ :

Theorem 1 (*Diacu and Santoprete [2002]*) *There are four heteroclinic orbits outside the collision manifold C that persists for all μ: $\gamma_{-\pi/2}$,γ_0 ,$\gamma_{\pi/2}$,γ_π connecting respectively $A^+_{-\pi/2}$ with $A^-_{-\pi/2}$, A^+_0 with A^-_0, $A^+_{\pi/2}$ with $A^-_{\pi/2}$, and A^+_π with A^-_π*





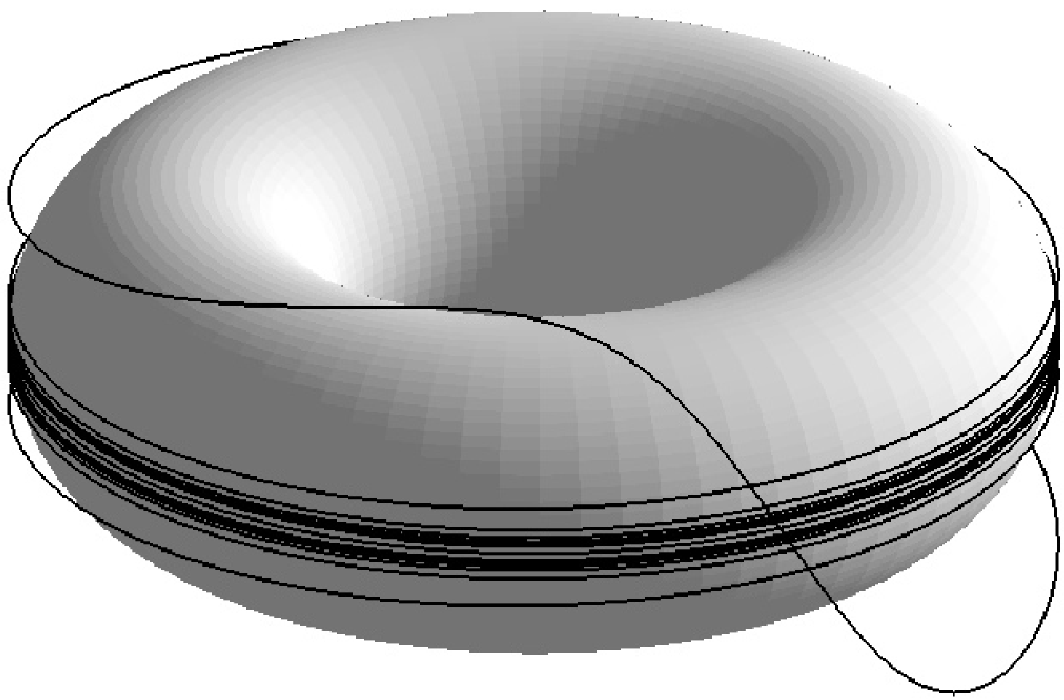
Proposition 1 *(Diacu and Santoprete [2002]) Let $W^s(p)$ and $W^u(p)$ denote the stable and unstable manifolds at the equilibrium point p. Then*

1. $\gamma_{-\pi/2} = W^u(A^+_{-\pi/2}) \cap W^s(A^-_{-\pi/2})$
2. $\gamma_{\pi/2} = W^u(A^+_{\pi/2}) \cap W^s(A^-_{\pi/2})$
3. $\gamma_0 \subset W^u(A^+_0) \cap W^s(A^-_0)$
4. $\gamma_\pi \subset W^u(A^+_\pi) \cap W^s(A^-_\pi)$

Proof: **The proof follows immediately from the dimension of the manifolds. If the manifolds are 1 dimensional we have equality, if they are 2 dimensional the other cases hold.**

Homoclinic Manifold

- **The orbits on the equator are nonhyperbolic. They have a positively and a negatively invariant sets.**
- **For $\mu = 1$ the sets coincide and form an homoclinic manifold.**



An orbit on the homoclinic manifold for $\mu = 1$

Homoclinic Manifold

- **Actually the equations that describe the manifold can be found explicitly: they have $u = \pm\sqrt{2b}$.**

$$R(\tau - \tau_0) = \frac{2}{2|h| + (\tau - \tau_0)^2}, \qquad R' = -\frac{4(\tau - \tau_0)}{(2|h| + (\tau - \tau_0)^2)^2}$$

and

$$V(\tau - \tau_0) = \frac{R'}{R} = -\frac{2(\tau - \tau_0)}{2|h| + (\tau - \tau_0)^2}.$$

Furthermore

$U(\tau - \tau_0) = \pm\sqrt{2b} = \omega$ **and** $\vartheta(\tau - \tau_0, \theta_0) = \Theta(\tau - \tau_0) - \theta_0$, **where** $\Theta(\tau - \tau_0) = \omega(\tau - \tau_0)$**. The Homoclinic orbit will be denoted as** $\chi = (R, V, \Theta, U)$**.**

Homoclinic Manifold

- ***What happens for $\mu > 1$?***
- ***To answer this question one can***
  - Write the anisotropy as a perturbation
  - Develop a Generalized Melnikov theory
  - Compute the Melnikov Integrals.

Perturbations

- If we choose the parameter $\epsilon = \mu - 1 \ll 1$, we can expand the equation of motion in powers of ϵ to obtain

$$\begin{cases} r' = rv \\ v' = 2r^2h + r - \epsilon(r/2\cos^2\theta) \\ \theta' = u \\ u' = \epsilon/2(r+2b)\sin 2\theta. \end{cases} \tag{2.1}$$

- The perturbaton can be viewed as a vector $(b_1, b_2, b_3, b_4) = (0, -(r/2\cos^2\theta), 0, 1/2(r+2b)\sin 2\theta)$.
- The energy relation becomes

$$u^2 + v^2 - 2r - 2b + \epsilon(r+2b)\cos^2\theta = 2r^2h.$$

Melnikov Method

There are transversal homoclinic solutions if and only if there exist τ_0^* and a θ_0^* such that

$$M_1(\tau_0^*,\theta_0^*) = M_2(\tau_0^*,\theta_0^*) = 0 \quad \textbf{and} \quad \left.\frac{\partial M_1}{\partial \tau_0}\frac{\partial M_2}{\partial \theta_0} - \frac{\partial M_1}{\partial \theta_0}\frac{\partial M_2}{\partial \tau_0}\right|_{\substack{\tau_0=\tau_0^* \\ \theta_0=\theta_0^*}} \neq 0,$$

where

$$M_1(\tau_0,\theta_0) = \int_{-\infty}^{+\infty} e^{-\int_{\tau_0}^{\tau} TrA(s)ds}\; R'(\tau-\tau_0)$$
$$\times b_2(\chi(\tau-\tau_0),\Theta(\tau-\tau_0)-\theta_0)\; d\tau,$$
$$M_2(\tau_0,\theta_0) = \int_{-\infty}^{+\infty} b_4(\chi(\tau-\tau_0),\Theta(\tau-\tau_0)-\theta_0)\; d\tau.$$

If the perturbation is periodic we get infinitely many intersections.

Melnikov Method

- Developing a Melnikov Theory in this case requires particular care because the periodic orbits are nonhyperbolic.
  - **The perturbation must not destroy the periodic orbits...**
  - **and their positively and negatively asymptotic sets.**
- In general even taking a perturbation that vanishes on the periodic orbit is not enough...
- In this particular case, even if the perturbation changes the periodic orbits, everything works out.

Homoclinics and Chaos

- Using the Melnikov Method we can prove the folowing

Theorem 2 *(Diacu and Santoprete [2002]) There is an infinite sequence of homoclinic intersections on the Poincaré section of the negatively and positively asymptotic sets of the periodic orbits at the equator of the collision manifold (possibly giving rise to a chaotic dynamics).*

- This also proves the existence of infinitely many orbits homoclinic to the "equator".
- To be precise this doesn't really prove the appearance of chaos, because the Smale-Birkhoff theorem does not apply.

Melnikov Integrals

Proof of the Theorem:

$$M_1(\tau_0,\theta_0) = \int_{-\infty}^{+\infty} \left[e^{-\frac{1}{2}\int_{\tau_0}^{\tau} V(s)ds} R(\tau-\tau_0)R'(\tau-\tau_0) \right.$$
$$\left. \times \cos^2(\omega(\tau-\tau_0)-\theta_0) \right] \, d\tau = 0$$

and

$$M_2(\tau_0,\theta_0) = \frac{1}{2}\int_{-\infty}^{+\infty} (R(\tau-\tau_0)+2b)\sin(2(\omega(\tau-\tau_0)-\theta_0)) \, d\tau = 0.$$

Computing the two integrals above using the method of residues one finds only one independent condition:

$$M_1 = M_2 = -\sin(2(\theta_0))\frac{\pi e^{-2\omega\sqrt{2|h|}}}{\sqrt{2|h|}}$$

There are simple zeroes when $\sin(2\theta_0) = 0$**,i.e. for** $\theta_0 = \pm k\pi/2$ **for** $k = 0, 1, 2...$**. This proves the Theorem.**

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Symmetric Periodic Orbits

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Symmetries

- The Symmetries of the equation of motion:

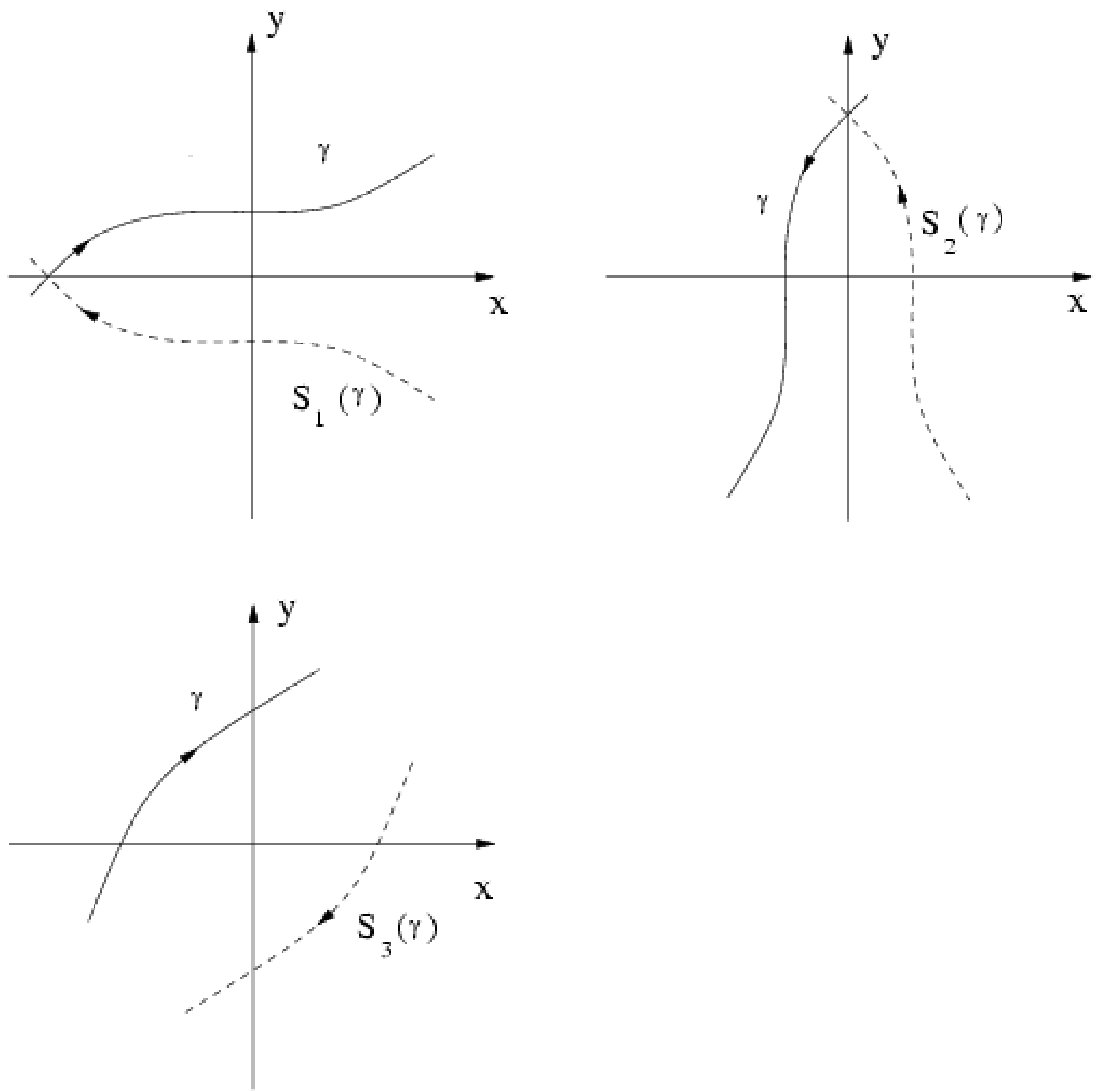
$$
\begin{aligned}
S_0(x,y,p_x,p_y,t) &= (x,y,-p_x,-p_y,-t),\\
S_1(x,y,p_x,p_y,t) &= (x,-y,-p_x,p_y,-t),\\
S_2(x,y,p_x,p_y,t) &= (-x,y,p_x,-p_y,-t),\\
S_3(x,y,p_x,p_y,t) &= (-x,-y,-p_x,-p_y,t),\\
S_4(x,y,p_x,p_y,t) &= (-x,y,-p_x,p_y,t),\\
S_5(x,y,p_x,p_y,t) &= (x,-y,p_x,-p_y,t),\\
S_6(x,y,p_x,p_y,t) &= (-x,-y,p_x,p_y,-t),
\end{aligned}
$$

  define a group isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ (Santoprete [2002]).
- These symmetries are useful to find periodic orbits (Santoprete [2002], Diacu and Santoprete [2002]).

Symmetries

- We will find S_1, S_2, S_3-symmetric periodic orbits using a variational methods.

Symmetries



The Symmetric orbits of $\gamma(t) : S_i(\gamma(t)$ for $i = 1, 2, 3$.

- Let $C^\infty([0,T],\mathbb{R}^2)$ be the space of T-periodic C^∞ cycles $f:[0,T]\to\mathbb{R}^2$.
- Define the inner products

$$\begin{aligned} \langle f,g\rangle_{L^2} &= \int_0^T f(t)\cdot g(t)dt, \\ \langle f,g\rangle_{H^1} &= \langle f,g\rangle_{L^2} + \langle \dot f,\dot g\rangle_{L^2}, \end{aligned} \tag{3.1}$$

  and let $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$ be the corresponding norms.
- The completion with respect to $\|\cdot\|_{L^2}$ is L^2.
- The completion with respect to $\|\cdot\|_{H^1}$ is the Sobolev space H^1 of all absolutely continuous T-periodic paths that have L^2 derivatives defined almost everywhere.

The spaces of Σ_i

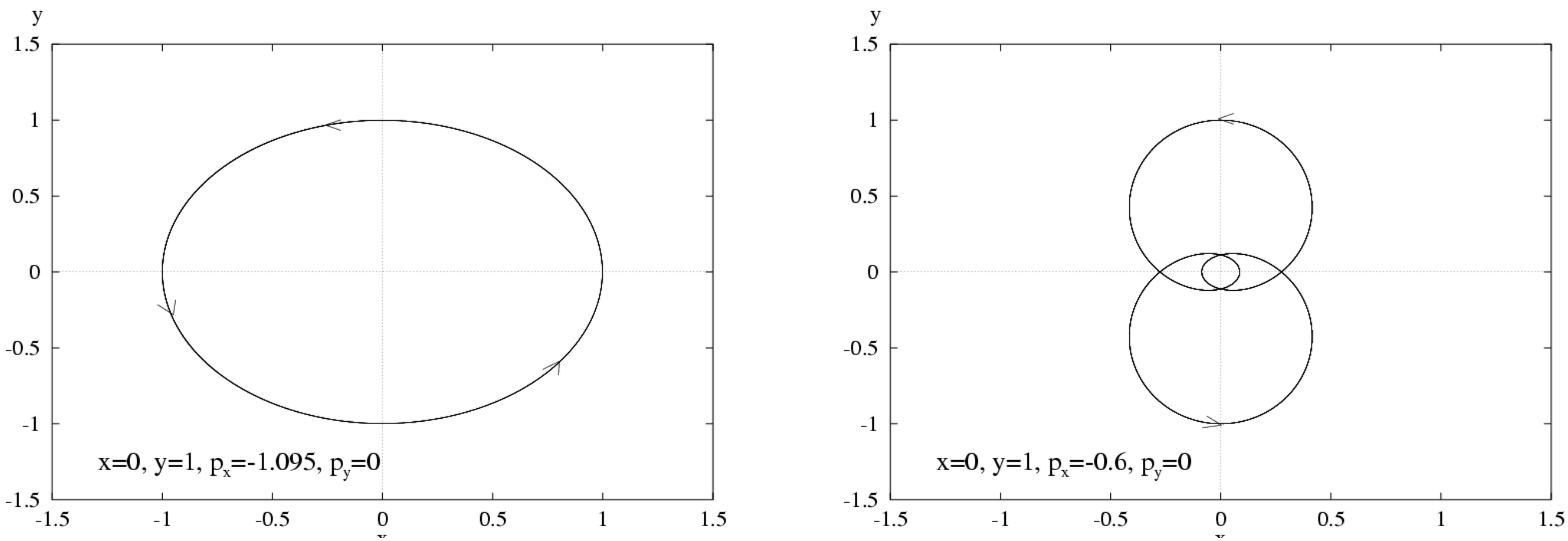
- Let $\Sigma_i([0,T],\mathbb{R}^2)$ denote the subset of H^1 formed by the S_i-symmetric paths, with $i \in \{0,1,2,3,4,5,6\}$.
- It is easy to see that each Σ_i is a subspace of H^1. Moreover we have that:

Lemma 1 ***Let*** H^1 ***be defined as above, then the subspaces*** Σ_i ***of*** S_i***-symmetric paths with*** $i = 0,1,2,3,4,5,6$ ***are complete with respect to the norm*** $\|\cdot\|_{H^1}$***, and therefore they are Sobolev spaces.***

Homotopy Classes

- We will say that a path in Σ_i is of class L_n, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, if its winding number about the origin of the coordinate system is n.
- Consider the sets $\overline{\Sigma}_i([0,T], \mathbb{R}^2 \backslash \{0\})$. Notice that they are open submanifolds of the spaces $\Sigma_i([0,T], \mathbb{R}^2)$ and that the family $(L_n)_{n \in \mathbf{Z}}$ provides a partition of those spaces into homotopy classes, also called components.

Homotopy Classes



Two periodic orbits of the Manev Problem. On the Left an S_3-symmetric orbit of class L_1. On the right S_1-symmetric orbit of class L_3

Hamilton Principle

- The Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) - U(\mathbf{q})$ of the anisotropic Manev problem given has the expression

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{\sqrt{\mu x^2 + y^2}} + \frac{b}{\mu x^2 + y^2},$$

- and the action integral along a path f from time 0 to time T is

$$A_T(f) = \int_0^T L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt.$$

- According to Hamilton's principle, the extremals of the functional A_T are solutions of the equations of motion.

The Variational Method

- We want to obtain periodic solutions by finding absolute minimizers of A.
- For this we will use a direct method of calculus of variation, namely the lower-semicontinuity method (Tonelli [1915]....Gordon [1975]) that gives extremals that belong to a Sobolev space.
- The lower-semicontinuity method provides only "weak" solutions of our problem.
- But it can be shown that the paths are regular enough to be classical solutions(Gordon [1975]).

Symmetrical Paths

- Consider the spaces Σ_i of S_i-symmetric paths for $i = 1, 2, 3$.
- It is not obvious that a collisionless minimizer in Σ_i is a periodic solution of the equation of motion.
- However, according to the principle of symmetrical criticality (Palais [1979]) this is actually true.
- Indeed, it can be proved that if f is a collision free path with $dA_t(f)(h) = 0$ for every $f \in \Sigma_i$, then $dA_T(f)(h) = 0$ for all $f \in H^1([0, T], \mathbb{R}^2)$ and thus f is a critical point in the bigger loop space H^1.

Noncompactness

- Before we can apply we want to exclude the possibility that:
  - **the minimizer is obtained when the bodies are at infinite distance from each other**
  - **the minimizer is a collision path.**
- The first problem is solved restricting ourselves to cycles that are not in the homotopy class L_0.

Noncompactness

- The second problem is solved by the following result:

Lemma 2 (*Diacu and Santoprete [2002]*) ***Any family Γ of non-simple homotopic cycles in $\overline{\Sigma}_i([0,T],\mathbb{R}^2\backslash\{0\})$ for $i=1,2,3$ on which $J(f)=\int_0^T \frac{1}{2}|\dot{\mathbf{q}}(t)|^2 dt$ and $E(f)=\int_0^T U(\mathbf{q}(t))dt$ are bounded, is bounded away from the origin.***

- The proof of this result follows from a similar theorem in a paper of Gordon if we remark that the anisotropic Manev potential is “strong” and that the Lagrangian is positive (same ideas already in Poincaré [1896])

Lower Semicontinuity

- Let $\mathcal{F} : X \to \mathbb{R}$ be a real valued function on a topological space X.
- $\mathcal{F}$ is lower semicontinuous (l.s.c.) if and only if $\mathcal{F}^{-1}(-\infty, a]$ is closed for every $a \in \mathbb{R}$
- in which case $\mathcal{F}$ is bounded below and attains its infimum on every compact subset of X.
- Moreover when X is Hausdorff then compact sets are necessarily closed and thus we have the following result.

Proposition 2 *(Gordon [1975]) Suppose $\mathcal{F} : X \to \mathbb{R}$ is a real valued function on an Hausdorff space X and*

$$\mathcal{F}^{-1}(-\infty, b] \quad \textit{is (weakly) compact for every real } b.$$

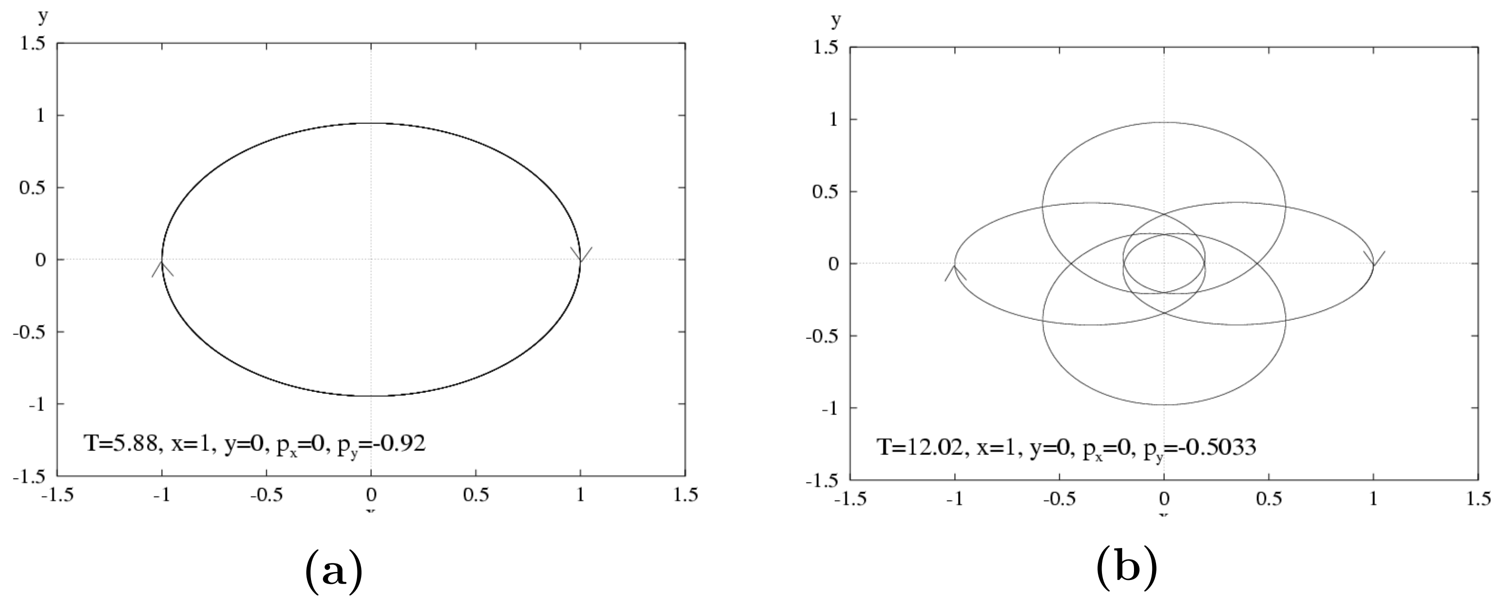
Then $\mathcal{F}$ is l.s.c., bounded below, and attains its infimum value on X.

- We can now prove the existence of symmetric periodic orbits:

Theorem 3 *(Diacu Santoprete [2002]) For any $T > 0$ and any $n = \pm 1, \pm 2, \pm 3, \ldots,$ there is at least one S_i-symmetric ($i =, 1, 2, 3$) periodic orbit of the anisotropic Manev problem that has period T and winding number n (i.e., belongs to the homotopy class L_n).*

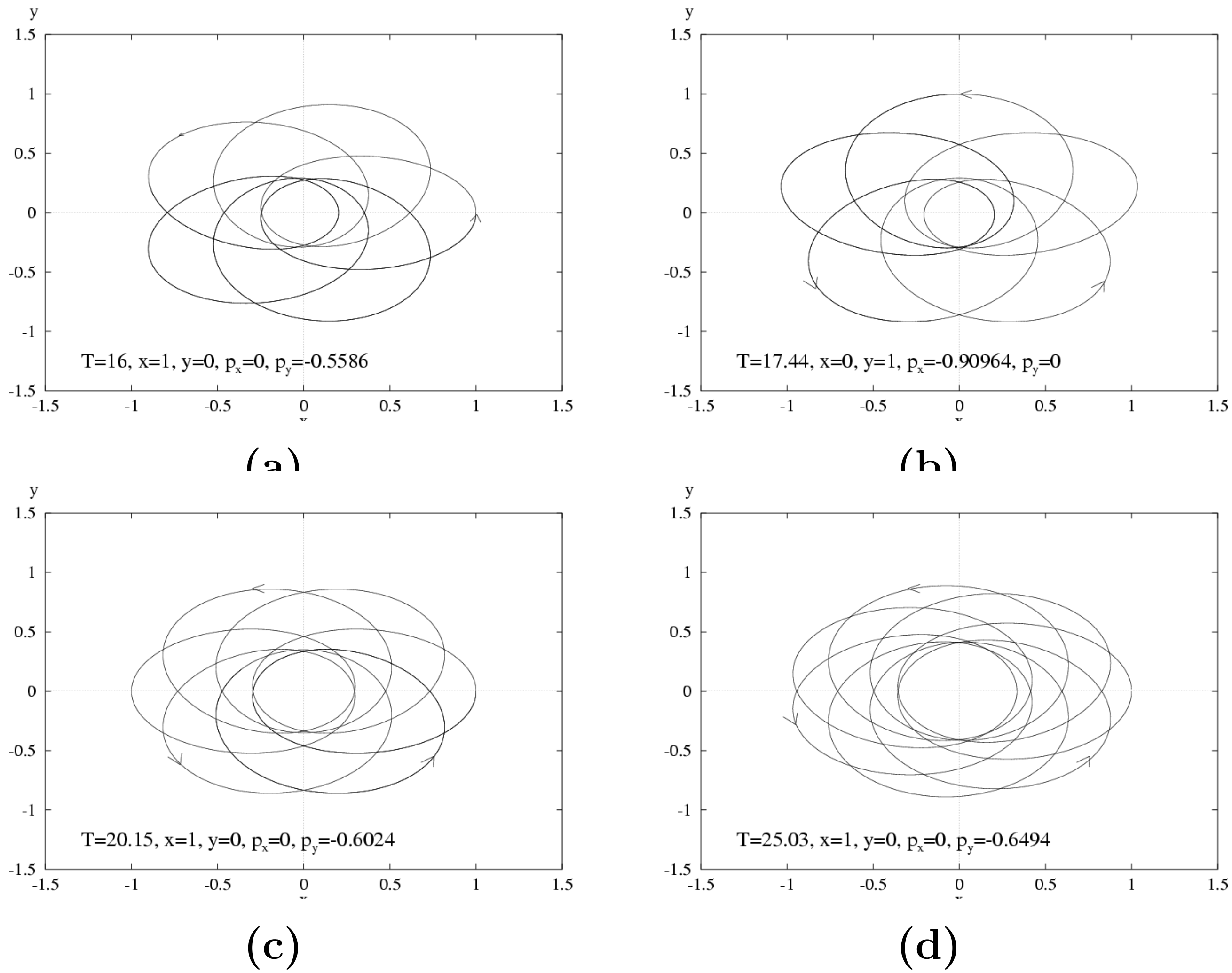
- Thus for each $T > 0$, each nonzero winding number we showed there is (at least) one S_i-symmetric orbit with $i = 1, 2, 3$.

- Some periodic orbits found numerically:



Two symmetric periodic orbits of the anisotropic Manev Problem with $\mu = 1.5$, $b = 0.1$. (a) A S_3-symmetric periodic orbit of class L_{-1}. (b) A S_3-symmetric periodic orbit of class L_{-5}.

Periodic Orbits



Four symmetric periodic orbits of the anisotropic Manev Problem with $\mu = 1.5$, $b = 0.1$. (a) A S_1-symmetric periodic orbit of class L_6. (b) A S_2-symmetric periodic orbit of class L_{-6}. (c) A S_3-symmetric periodic orbit of class L_{-7}. (d) A S_1-symmetric periodic orbit of class L_{-8}.

Conclusions

- We studied the the collision orbits and the mechanism responsible for the appearance of chaos.
  - **We found that the collision orbits create an intricate (chaotic) pattern (see Devaney [1978] for a similar result for the AKP)**
  - **We developed a new Melnikov-type technique to detect chaos.**
- We found that the symmetries have important consequences
  - **Even for large perturbations the symmetric periodic orbits are not destroyed.**

Bibliography

- **Chenciner, A. and Montgomery, R.(2000)** *A remarkable periodic solution of the three-body problem in the case of equal masses*. **Ann. Math. 152, 881-901.**
- **Chenciner A.: Action minimizing periodic orbits in the Newtonian n-body problem, to appear in the** *Proceedings of the Evanston Conference dedicated to Don Saari (Dec. 15-19, 1999)*, **to appear.**
- **Craig S., Diacu F., Lacomba E. A., and Perez E. (1999)** *On the anisotropic Manev problem*. **J. Math. Phys. 40, 1359-1375.**
- **Devaney R.L. (1978)** *Collision Orbits in the Anisotropic Manev Problem*. **Invent. Math. 45, 221-251.**
- **Diacu F. and Santoprete M. (2002)** *On the Global Dynamics of the Anisotropic Manev Problem*, **preprint.**
- **Gordon, W.B. (1975).** *Conservative Dynamical Systems In-*

volving Strong Forces **Transactions of the American Mathematical Society, 204, 113-135.**

- **Palais R. (1979).** *The Principle of Symmetrical Criticality* **Comm. Math. Phys.,69, 19-30.**
- **Poincaré, H. (1896)** *Sur les solutions périodiques et le principe de moindre action*.**Comptes rendus de l'Academie de Sciences, 123, 915-918 .**
- **Santoprete M. (2002)** *Symmetric periodic solutions of the anisotropic Manev problem.* **J. Math. Phys. 43, 3207-3219.**
- **Tonelli L. (1915) Sur une méthode directe du calcul des variations Circ. Matem. Palermo, 39, 233-263.**

The End

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