New Phenomena Associated With Homoclinic Tangencies

by

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Summary:

1. Def of homoclinic points, basic properties

2. the homoclinic relation: on hyperbolic sets

3. generic consequences of homoclinic tangencies

   (a) C^r, $r > 1$, homoclinic set (h-closure) $\subset Closure(sinks) \bigcup Closure(sources)$; *homoclinic set has Hausdorff dim = 2, no SRB measure, no principal symbolic extension*

   (b) C^1 area preserving: elliptic periodic points dense; *no symbolic extensions at all; homoclinic set has Hausdorff dim = 2.*

M = compact surface

$\mathcal{D}^r(M)$ = space of C^r diffeomorphisms of M with the uniform C^r topology, $r \geq 1$.

$\mathcal{D}^r_\omega(M)$ = space of volume preserving diffeomorphisms

Let $f \in \mathcal{D}^r(M)$. $\Lambda = \Lambda(f)$ be a compact invariant hyperbolic basic set

That is,

- $f(\Lambda) = \Lambda$, $\exists$ neighborhood $U \supset \Lambda$ with $\bigcap_n f^n(U) = \Lambda$
- $f \mid \Lambda$ is *topologically transitive* , periodic points dense
- $\exists$ splitting $TM_\Lambda = E^u \oplus E^s$ and constants $C > 0,\ \lambda > 1$ such that

$$\mid Df^n_x \mid E^s_x \mid \leq C\lambda^{-n},\ \mid Df^{-n}_x \mid E^u_x \mid \leq C\lambda^{-n} \text{ for } n \geq 0,\ x \in \Lambda$$

Properties of hyperbolic basic sets:

- persistence and conjugate stability:

– For g C^1 near f, $\bigcap_n g^n(U) = \Lambda(g)$ is a hyperbolic basic set
– $\exists$ a homeomorphism $h : \Lambda(g) \to \Lambda(f)$ such that $fh = hg$.

- invariant manifolds: for $x \in \Lambda(f)$,

$$W^u(x) = \{y \in M : d(f^{-n}y, f^{-n}x) \to 0,\ n \to \infty\}$$

$$W^s(x) = \{y \in M : d(f^n y, f^n x) \to 0,\ n \to \infty\}$$

are injectively immersed C^r curves in M, depending continuously on compact parts

Definitions

$\hat{W}^u(x) = W^u(x) \setminus \{x\}, \hat{W}^s(x) = W^s(x) \setminus \{x\}$

Homoclinic Point: $q \in \hat{W}^u(x) \bigcap \hat{W}^s(y)$,
for $x, y \in \Lambda$, hyperbolic basic set

Transverse Homoclinic Point: $\hat{W}^u(x) \pitchfork_q \hat{W}^s(y)$

Homoclinic Tangency: $\hat{W}^u(x)$ tangent to $\hat{W}^s(y)$ at q

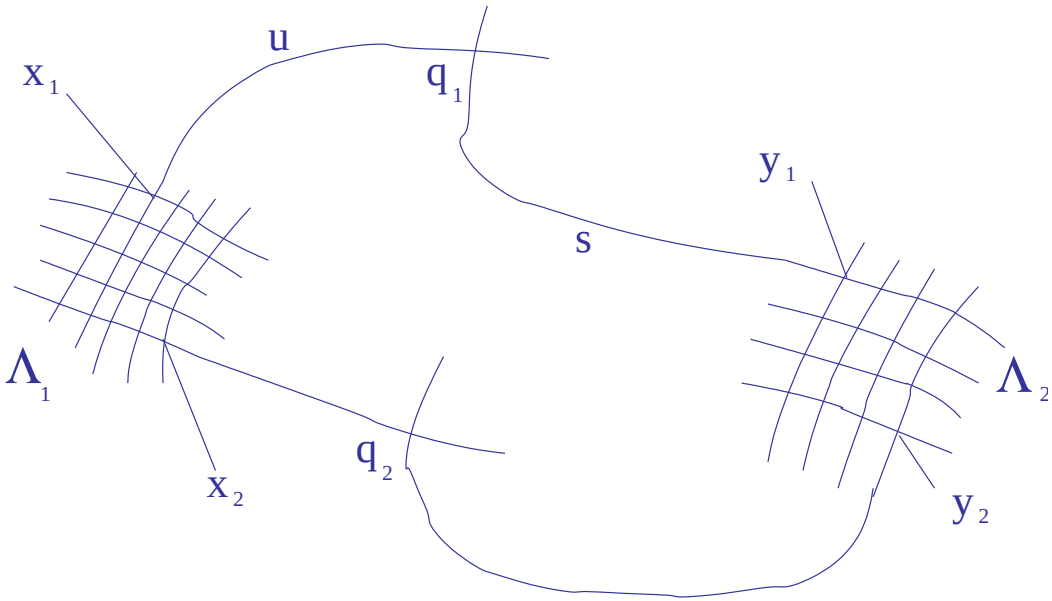
Homoclinic Relation (h-relation): $\Lambda_1 \sim \Lambda_2$ if

$\exists\ x \in \Lambda_1,\ y \in \Lambda_2,\ q \in \hat{W}^u(x) \bigcap \hat{W}^s(y)$, transverse, and vice-versa

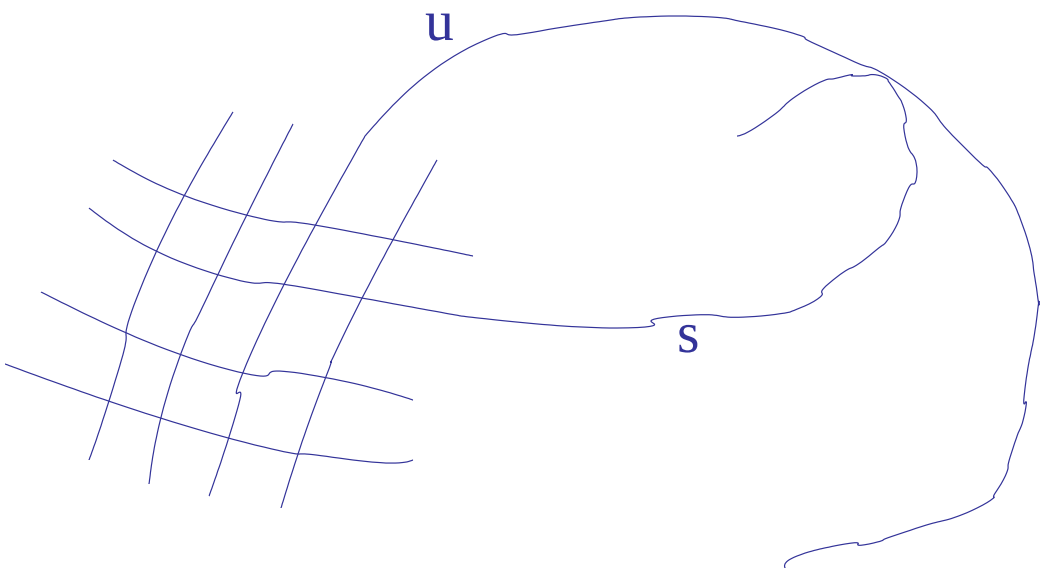
Fact: $\sim$ is an equivalence relation on the set of hyperbolic basic sets;

Homoclinic Class (h-class): equivalence class of $\sim$

Homoclinic Set (h-closure):
Closure of the union of an equivalence class of $\sim$



$$\Lambda_1 \sim \Lambda_2$$



Homoclinic tangency

Properties of Homoclinic Sets:

- Topologically transitive with periodic points dense

- If hyperbolic, then hyperbolic basic set; i.e. has adapted neighborhood

- If $\Lambda_1 \sim \Lambda_1$ are hyperbolic basic sets, then there is another hyperbolic basic set Λ_3 such that

$$\Lambda_3 \supset \Lambda_1 \bigcup \Lambda_2 \text{ and } \Lambda_1 \bigcup \Lambda_2 \subset \Lambda_3.$$

Let $H(\Lambda(f))$ be the homoclinic set of a hyperbolic basic set. If $\Lambda(f)$ is non-trivial (i.e., infinite), then $H(\Lambda(f)) \neq \emptyset$.

- the map $f \to H(\Lambda(f))$ is lowersemicontinuous

The Palis Conjecture

Hyperbolic Diffeomorphisms:

Def: f is *hyperbolic* if $\mathcal{R}(f)$ is a hyperbolic set (Axiom A + No Cycles).

Def: *Chain Recurrent Set of f*, $\mathcal{R}(f)$: Set of x such that for every $\epsilon > 0$ there is a sequence $x = x_0, x_1, \ldots, x_n = x$ such that $d(f(x_{i+1}), x_i) < \epsilon$ for $0 \leq i < n$. ($\epsilon-$precision periodic orbit for each $\epsilon > 0$).

- $\mathcal{R}(f)$ is a compact invariant set.
- Points in $\mathcal{R}(f)$ exhibit *very mild* recurrence properties.

Global Stability Property:

Def: f is *chain stable* if there is a neighborhood $\mathcal{N}(f) \subset \mathcal{D}^1(M)$ such that $g \in \mathcal{N}(f)$ implies $(g, \mathcal{R}(g))$ is topologically conjugate to $(f, \mathcal{R}(f))$.

Theorem. f is *chain stable* if and only if f is *hyperbolic*.

Def: f has *persistent homoclinic tangencies* if there is a neighborhood $\mathcal{N}(f)$ in $\mathcal{D}^r(M)$ such that any $g \in \mathcal{N}(f)$ has homoclinic tangencies. ▮

Palis Conjecture: There is a dense subset $\mathcal{B}$ in $\mathcal{D}^r(M)$ such that if $f \in \mathcal{B}$, then either f is hyperbolic or f has a homoclinic tangency. ▮

True for $r = 1$ (Pujals-Samborino); no known examples of persistence of homoclinic tangencies for C^1 surface diffeomorphisms. ▮

Extended Palis Conjecture: There is a residual subset $\mathcal{B}$ in $\mathcal{D}^r(M)$ such that if $f \in \mathcal{B}$, then any homoclinic set is either uniformly hyperbolic or has a homoclinic tangency. ▮

Old Results: ▮

Theorem 1. Fix $r > 1$.

- If f has a homoclinic tangency, then every C^r neighborhood of f contains diffeomorphisms with persistent homoclinic tangencies.
- C^r generically in $\mathcal{D}^r(M)$, each homoclinic set with tangencies is in the closure of the periodic sinks or sources.

Theorem 2. Fix $r = 1$. C^1 generically in $\mathcal{D}^1_\omega(M)$

either f is hyperbolic (Anosov) or

- elliptic periodic points are dense, and
- each homoclinic set has maximal Hausdorff dimension (HD $= 2$).

A recent theorem

Theorem 3. (Arnaud-Bonatti-Crovisier (2001) There is a residual set $\mathcal{B}$ in the space of C^1 volume preserving or symplectic diffeomorphisms on any compact manifold such that if $f \in \mathcal{B}$, then all of M is a single homoclinic class.

Remark.

1. This implies that f as above is topologically transitive. What about ergodic?

2. This says that KAM fails in a spectacular way in C^1 area preserving diffeomorphisms

SRB measures and homoclinic tangencies

Def: An SRB measure for f is an ergodic invariant probability measure with non-zero Lyapunov exponents and absolutely continuous conditional measures on unstable manifolds.

Precise definition not needed:

Some properties of SRB measure μ:

(a) ergodic: $f^{-1}(E) = E \Longrightarrow \mu(E) = 0$ or 1

(b) non-zero Lyapunov exponents:

$$\int \log Df(x) d\mu(x) > 0, \quad \int \log Df^{-1}(x) d\mu(x) > 0,$$

(c) there is a set $B(\mu)$ of positive Lebesgue measure such that for $x \in B(\mu)$, and continuous ϕ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu,$$

(d) for $\mu-$almost all points x, the sets

$$W^u(x, f) = \{y : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}y, f^{-n}x) < 0\}$$

and

$$W^s(x, f) = \{y : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{n}y, f^{n}x) < 0\}$$

are injectively immersed curves (Pesin stable and unstable manifolds)

(e) for $\mu-$almost all x, $W^u(x, f)$ is contained in the support of μ.

Def: A periodic point p of f of period τ is called *dissipative* if $| \, detDf^{\tau}(p) \, | < 1$. ▮

Theorem 3. Let $r > 1$. There is a residual subset $\mathcal{B}$ of $\mathcal{D}^r(M)$ such that if $f \in \mathcal{B}$ and Λ is a homoclinic set for f containing a homoclinic tangency and a dissipative periodic point, then ▮

(a) *no* SRB measure for f can be supported on Λ, and ▮
(b) $HD(\Lambda) = 2$. ▮

Remark 1: Statement 1 in the theorem holds generically even in the Henon family. ▮

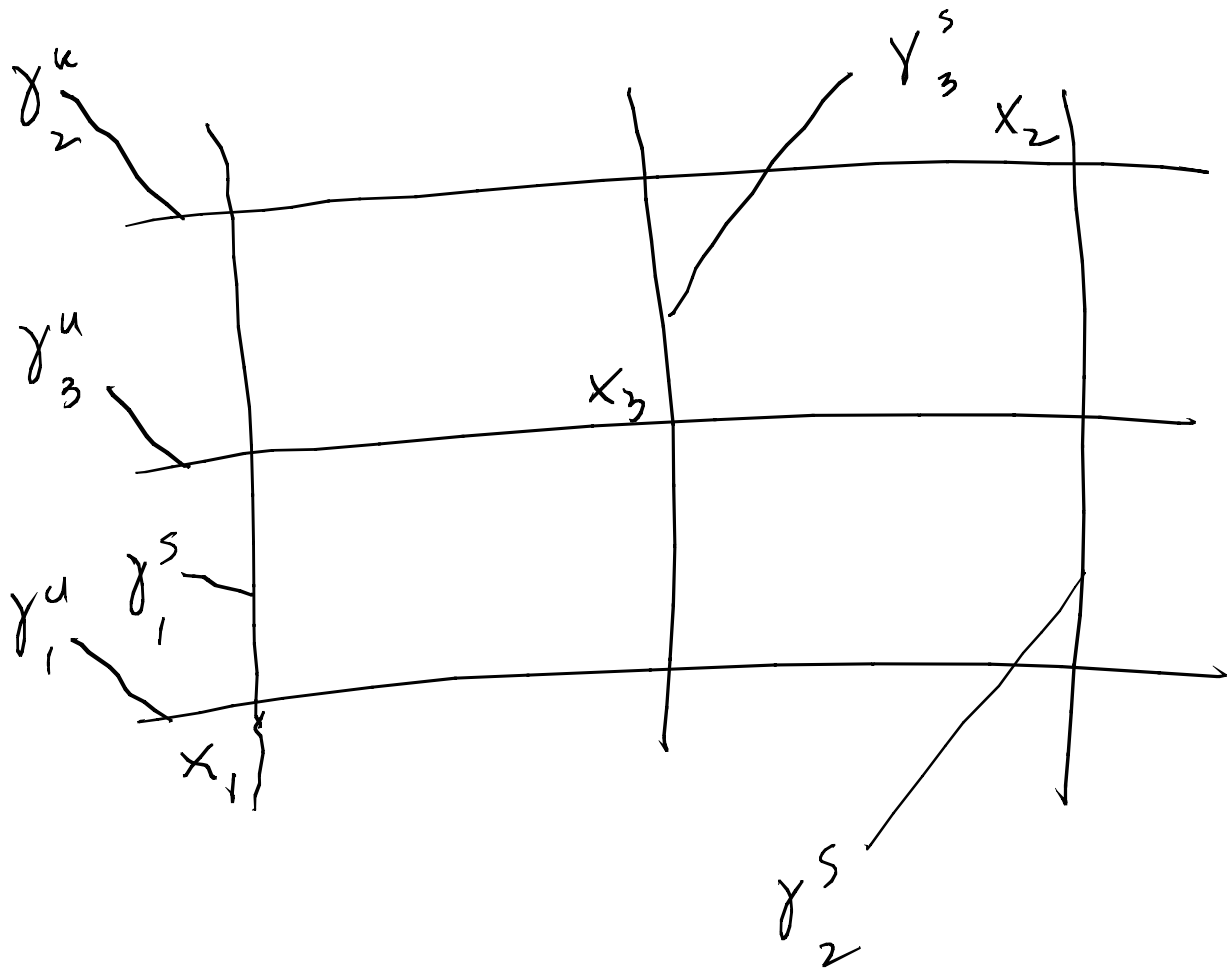
Remark 2: The Extended Palis Conjecture (C^r, $r > 1$) implies that generically any SRB measure which is $h-$related to a dissipative periodic orbit must be supported on a uniformly hyperbolic set. ▮

Remark 3: Analogous to the 1-dim family $f_r(x) = rx(1-x)$.

Idea of Proof:

• Recall that an SRB measure has full unstable manifolds of almost all points in its support. From this, results of Katok, and the Palis Inclination Lemma, one gets that every unstable manifold in the homoclinic class of an SRB measure is contained in its support.

See the following figure


γ^k_2
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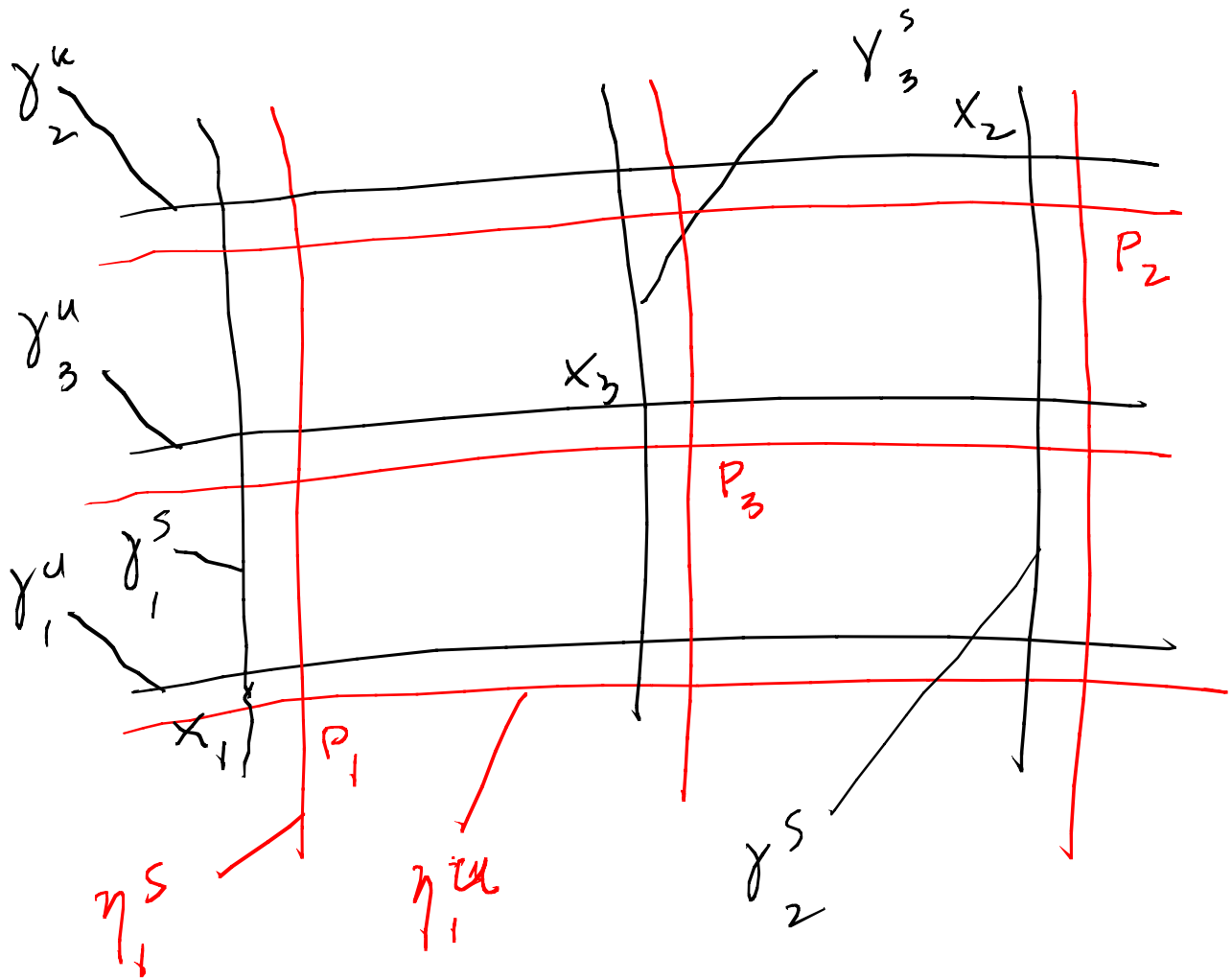
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γ_3^s
X_2
P_2
γ_3^u
X_3
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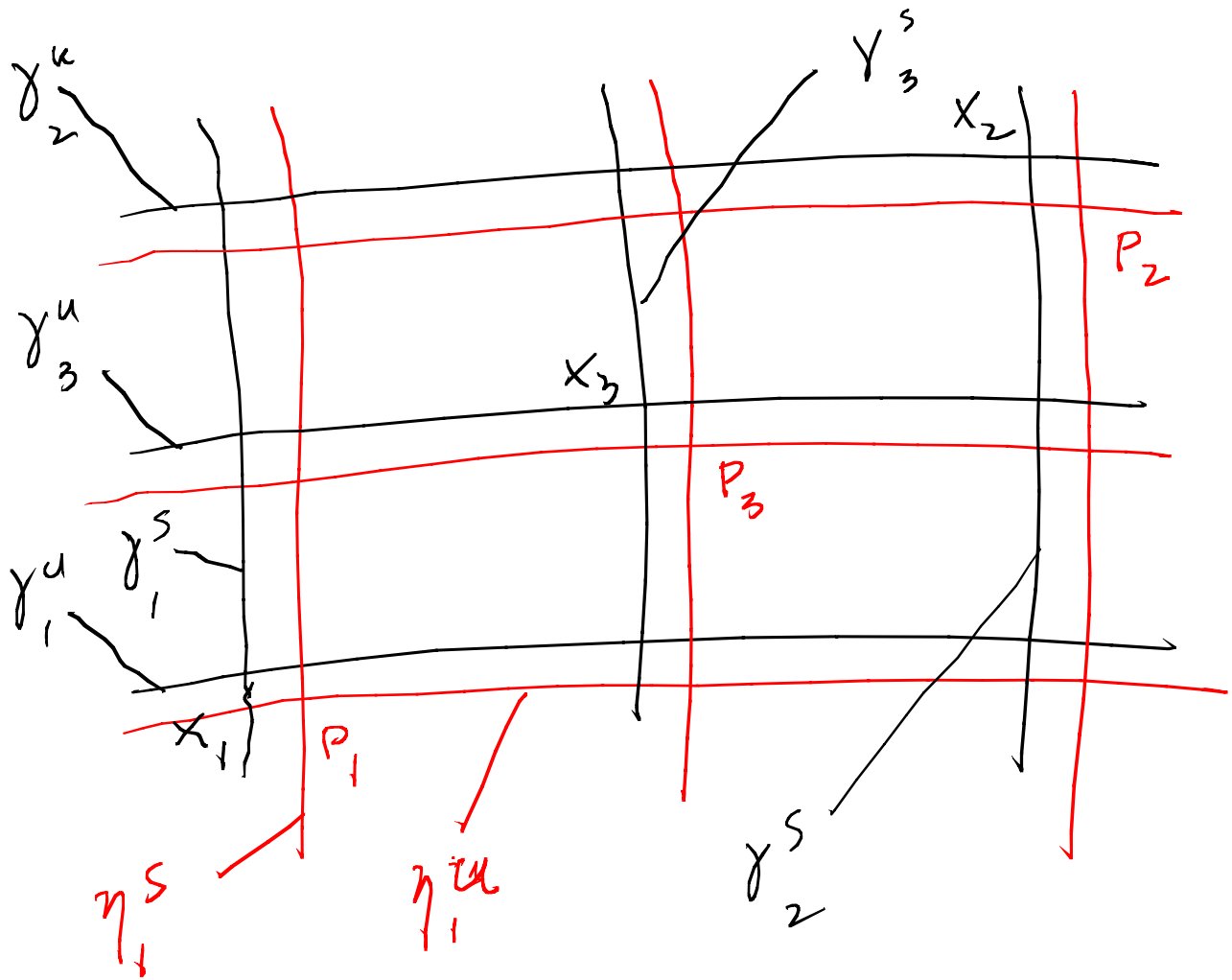
η_1^s
η_1^u
γ_2^s

Katok

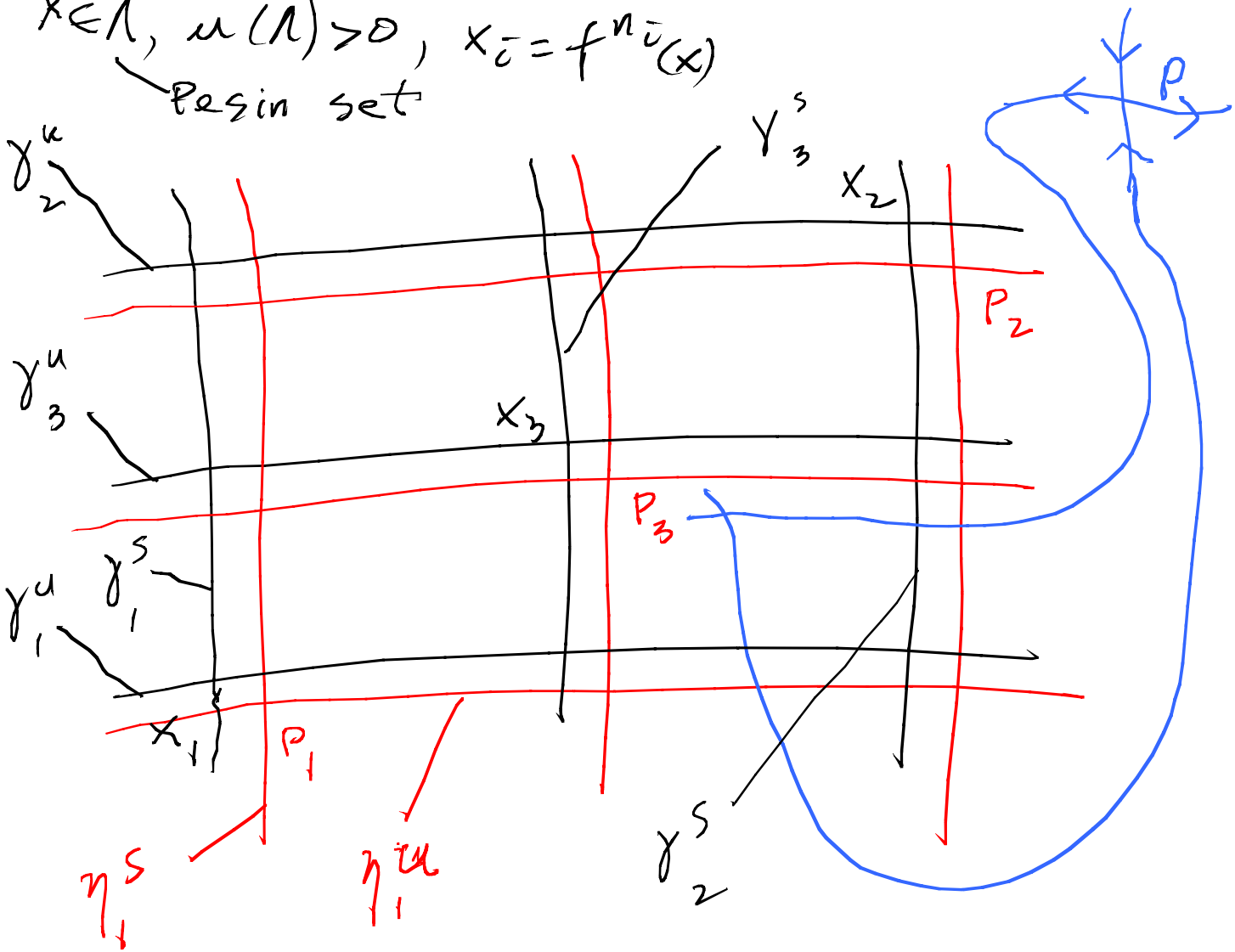


Katok

$\gamma_i^u, \eta_i^u \subset \mathrm{supp}(\mu)$

$x \in \tilde{\Lambda}$, $\mu(\tilde{\Lambda}) > 0$, $x_i = f^{n_i}(x)$

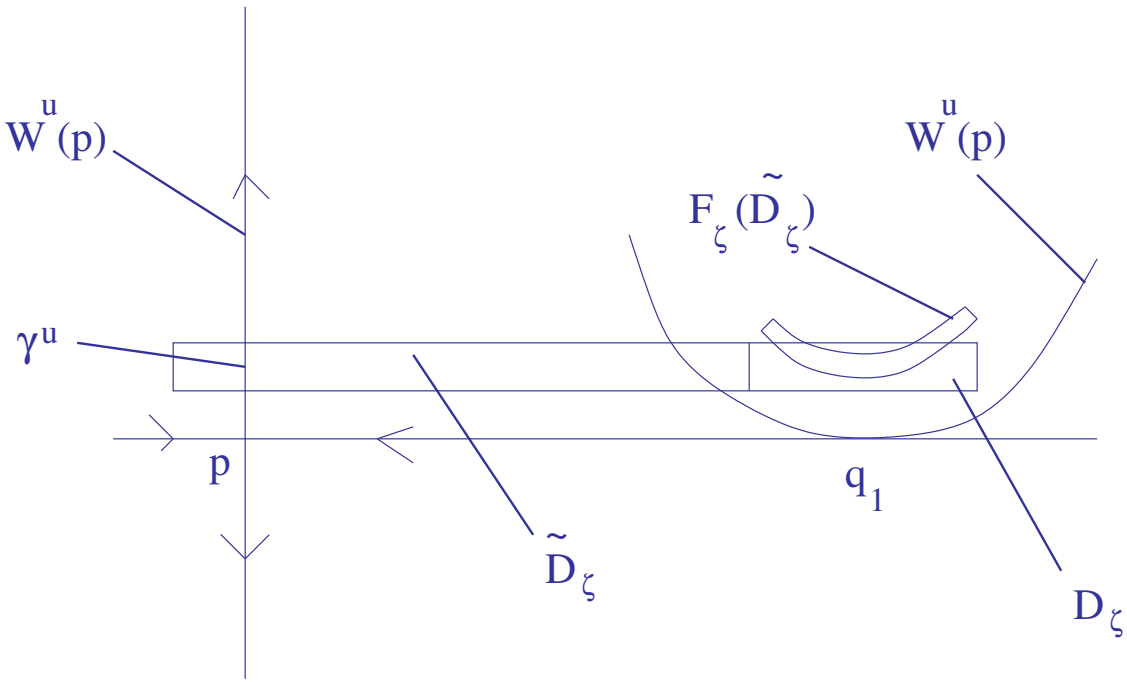
Pesin set



Katok

$\gamma_i^u, \eta_i^u \subset \mathrm{supp}(\mu)$

• If p is dissipative and has a homoclinic tangency, then f can be perturbed so that some interval in $W^u(p)$ is in the stable manifold of a sink.



Hausdorff dimension results:

Use result of Manning-McCloskey:
If Λ is a hyperbolic saddle set such that there is an invariant measure μ such that

$$h_\mu(f) = \lambda \int \log |\, Df \,| d\mu,$$

Then,

$$HD^u(\Lambda) \geq \lambda.$$

Let $\epsilon > 0$

Let Λ be a homoclinic set containing a tangency.

(a) Perturb to create interval of tangencies (Gonchenko-Silnikov-Turaev, Kaloshin)

(b) Perturb to get $\Lambda_1 \subset \Lambda$ with $HD^u(\Lambda_1) > 1 - \epsilon$

(c) Perturb to get $\Lambda_2 \subset \Lambda$ with $HD^s(\Lambda_2) > 1 - \epsilon$

(d) Get $\Lambda_3 \subset \Lambda$ containing $\Lambda_1 \bigcup \Lambda_2$

So, $HD(\Lambda_3) > 2 - \epsilon$, depends continuously,

$\implies HD(\Lambda) = 2$, generically