CDS 213 - Robust Control
Homework # 2

Date Given: February 5th, 2004
Date Due: February 19th, 2004

P1. Prove corollary 6.6 in DP p. 204. Also, prove that H is in the domain of the Riccati operator if and only if (A, B) is stabilizable and (C, A) has no unobservable modes on the imaginary axis. Also prove that $\mathrm{Ker}(X) = 0$ if and only if (C, A) has no stable unobservable modes (if you get stuck, and only after thinking about it for some reasonable time, you are allowed to look into theorem 12.4 in Zhou).

P2. Do question 6.7, DP, p. 214.

P3. Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ be given and consider the Sylvester Equation

$$AX + XB = C$$

for an unknown matrix $X \in \mathbb{R}^{m\times n}$. Let

$$M = \begin{bmatrix} B & 0 \\ C & -A \end{bmatrix}, \qquad N = \begin{bmatrix} B & 0 \\ 0 & -A \end{bmatrix}.$$

1. Let the columns of $\begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{C}^{(n+m)\times n}$ be the eigenvectors of M associated with the eigenvalues of B and suppose U is nonsingular. Show that

$$X = VU^{-1}$$

solves the Sylvester Equation. Every solution of the Sylvester Equation can be written in the above form.

2. Show that the Sylvester Equation has a solution if and only if M and N are similar.

P4. Let $A \in \mathbb{R}^{n\times n}$. Show that

$$P(t) = \int_0^t e^{A^*\tau} Q e^{A\tau} d\tau$$

satisfies

$$\dot{P}(t) = A^*P(t) + P(t)A + Q, \qquad P(0) = 0.$$

P5. Let $A \in R^{n\times n}$, $R = R^*$, $Q = Q^*$. Define

$$H = \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}.$$

Let

$$\Theta(t) = \begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} = e^{Ht}.$$

Show that

$$P(t) = (\Theta_{21} + \Theta_{22}P_0)(\Theta_{11} + \Theta_{12}P_0)^{-1}$$

is the solution to the following differential Riccati Equation

$$-\dot{P}(t) = A^*P(t) + P(t)A + PRP + Q, \qquad P(0) = P_0.$$

P6. Suppose D has full column rank and let $R = D^*D > 0$; Prove that the following statements are equivalent:

1. $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all ω,
2. $((I - DR^{-1}D^*)C, A - BR^{-1}D^*C)$ has no unobservable modes on the $j\omega$ axis.

In the case $D^*C = 0$ what does the second condition simplify to? Combining this lemma with the result in Problem (1), state a corollary, giving the form of the Hamiltonian H.

P7. Let a dynamical system G be described by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \qquad x(0) = x_0 \text{ given but arbitrary} \\ z &= C_1 x + D_{12} u \\ y &= x \end{aligned}$$

and suppose the system parameter matrices satisfy the following assumptions:

1. (A, B_1) and (A, B_2) are stabilizable and (C_1, A) is detectable,
2. $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

Solve the following problem:

$$\min_{K(s) \text{ stabilizing}} ||\underline{S}(\hat{G}, \hat{K})||_2.$$