Poincaré-Lindstedt Method

CDS140B

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1 Periodic Solutions of Autonomous 2nd Order Equations

Consider

$$\ddot{x} + x = \epsilon f(x, \dot{x}, \epsilon).$$

Assumptions:

- periodic solutions exist for small, postive ϵ;
- requirements of the Poincaré expansion thoerem have been satisfied.

Basic Ideas:

- Non-linear perturbation terms alters the period and the frequency of the unperturbed linear problem and we now have $T(\epsilon)$ and $\omega(\epsilon)$;
- Introduce a new time-like variable θ such that the periodic solution is 2π-periodic

$$\theta = \omega t, \quad \omega^{-2} = 1 - \epsilon\eta(\epsilon)$$

- Rewrite the equation using θ as the independent variable

$$x'' + x = \epsilon[\eta x + (1 - \epsilon\eta) f(x, (1 - \epsilon\eta)^{-1/2} x', \epsilon)] = \epsilon g(x, x', \epsilon, \eta)$$

  with initial values $x(0) = a(\epsilon), x'(0) = 0$.

- If the Jacobian of the periodicity conditions is non-zero, the corresponding periodic solution of the perturbed equation can be represented by the convergent series

$$x(\theta) = a(0) \cos\theta + \sum_{n=1}^{\infty} \epsilon^n \gamma_n(\theta).$$

- To determine $\gamma_n(\theta)$, we substitute the series into the rescaled equation, collect terms which are coefficients of equal powers of ϵ and produces equations for $\gamma_n(\theta)$.

  Since

$$a = \sum \epsilon^n a_n, \quad \eta = \sum \epsilon^n \eta_n, \quad x(0) = a(0) + \sum \epsilon^n \gamma_n(0),$$

  the initial conditions for equations of $\gamma(\theta)$ are $\gamma_n(0) = a_n, \gamma_n'(0) = 0$.

- The periodic conditions will allow us to determine all the necessary constants a_n, η_n, etc. Also, by using the relation $\omega^{-2} = 1 - \epsilon\eta_0 - \epsilon^2\eta_1 - \epsilon^3\eta_2 - \cdots$, we find

$$\omega = 1 + \tfrac{1}{2}\epsilon\eta_0 + \epsilon^2(\tfrac{1}{2}\eta_1 + \tfrac{3}{8}\eta_0^2) + \cdots$$

For the period, we have

$$T = \frac{2\pi}{\omega} = 2\pi[1 - \tfrac{1}{2}\epsilon\eta_0 - \epsilon^2(\tfrac{1}{2}\eta_1 + \tfrac{1}{8}\eta_0^2) - \cdots].$$

Remarks on Perodicity Conditions.

- The initial value a and the parameter η, which determines the unknown frequency, have to be chosen such that we obtain a $2\pi-$periodic solution in θ.

  By variation of constants, the intial value problem is equivalent to the following integral equation

$$x(\theta) = a\cos\theta + \epsilon\int_0^\theta \sin(\theta-\tau)g(x(\tau), x'(\tau), \epsilon, \eta)d\tau.$$

  For periodic solution, $x(\theta) = x(\theta + 2\pi)$ yields the periodicity condition (10.3):

$$\begin{aligned}\int_0^{2\pi} \sin\tau g(x(\tau), x'(\tau), \epsilon, \eta)d\tau &= 0\\ \int_0^{2\pi} \cos\tau g(x(\tau), x'(\tau), \epsilon, \eta)d\tau &= 0.\end{aligned}$$

- The periodic solutions (10.3) depends on ϵ but also on a and η, so it can be viewed as a system of two equations with two unknowns, a and η. According to the implicit function theorem this system is uniquely solvable in a neighborhood of $\epsilon = 0$ if the corresponding Jacobian (10.4) does not vanish.

$$|\frac{\partial(F_1, F_2)}{\partial(a, \eta)}| \neq 0$$

  If condition (10.4) has been satisfied, $a(\epsilon)$ and $\eta(\epsilon)$ can be Taylor expanded w.r.t. ϵ.

- From equation (10.2) we find the system (10.3) with $\epsilon = 0$.

$$\begin{aligned}\int_0^{2\pi} \sin\tau f(a(0)\cos\tau, -a(0)\sin\tau, 0)d\tau &= 0\\ \pi\eta(0) + \int_0^{2\pi} \cos\tau f(a(0)\cos\tau, -a(0)\sin\tau, 0)d\tau &= 0\end{aligned}$$

- Applying (10.4) to the above system (10.5) we find the condition (10.6)
  (notation: $f = f(x, y, \epsilon)$)

$$a(0)\int_0^{2\pi}[\tfrac{1}{2}\sin 2\tau\frac{\partial f}{\partial x}(a(0)\cos\tau, -a(0)\sin\tau, 0) + \sin^2\tau\frac{\partial f}{\partial y}(a(0)\cos\tau, -a(0)\sin\tau, 0)]d\tau \neq 0.$$

- If $\epsilon = 0$, all solutions are periodic. Condition 10.4 and 10.6 is condition for the existence of an isolated periodic solution which branches off for $\epsilon > 0$. If, however, there exists for $\epsilon > 0$ a continuous family of periodic solutions, a one-parameter family depending on $a(\epsilon)$, then the above condtion will not be satisfied. If we know aprior that this family of periodic solutions exists, then we can of course still apply the P-L method.

2 Approximation of period solutions on long time-scale

Remark: When computing a periodic solution of an equation with the PL method, the period and other characteristic quantities (amplitude, phase) can be approximated with arbitrarily good precision. One of the consequences is that we can find approximations which are valid on an interval of time which is much longer than the period. See section 10.2 of Verhulst.

3 Periodic solutions of equations with forcing terms

Remark: The theory of nonlinear differential equations with inhomogeneous time-dependent terms, which respresent oscillating systems with exciting forces, turns out to be very rich in phenomena. An important prototype problem, the forced Duffing equation, has been studied in section 10.3 of Verhulst.

4 The existence of periodic solutions

Theorem 10.1 Consider equation (10.1). If the conditons of the Poincaré expansion theorem have been satisfied and if the periodic condition (10.3) and the uniqueness condition (10.4) have been met, then there exists a periodic solution which can be represented by a conergent power series in ϵ for $0 \leq \epsilon < \epsilon_0$.

Example (van der Pol equation): Consider

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}.$$

- It has one periodic solution for small, postive ϵ.