Lecture 1B: Introdction to Perturbation Theory

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Winter, 2004

1 Basic Ideas

- **Perturbed and unperturbed problems:** To find approixmated solutions to a perturbed problem, one will always assume that one have sufficient knowledge of the solutions of the unperturbed problem.
- **Time scale:** In construting approximated solutions, one has to indicated on what interval of time (time-scale) one is looking for an approximation.

Example (Duffing equation):

$$\ddot{q} + q + \epsilon q^3 = 0.$$

2 Naive Expansion

Theorem 9.1: Consider the initial value problem

$$\dot{x} = f_0(t,x) + \epsilon f_1(t,x) + ... + \epsilon^m f_m(t,x) + \epsilon^{m+1} R(t,x,\epsilon)$$

with $x(t_0) = \eta$ and $|t - t_0| \leq h, x \in D \subset R^n, 0 \leq \epsilon \leq \epsilon_0$. Assume that in this domain

1. $f_i(t,x), i = 0, ..., m$ continuous in t and x, $(m + 1 - i)$ times continuously differentiable in x;
2. $R(t,x,\epsilon)$ continuous in t, x and ϵ, Lipschitz-continous in x.

Subsitituting in the equation for x the formal expansion

$$x_0(t) + \epsilon x_1(t) + ... + \epsilon^m x_m(t),$$

Taylor expanding w.r.t. powers of ϵ, equating corresponding coefficients and applying the initial values $x_0(t_0) = \eta, x_i(t_0) = 0, i = 1, ..., m$ produces an approximation of $x(t)$:

$$||x(t) - (x_0(t) + \epsilon x_1(t) + ... + \epsilon^m x_m(t))|| = O(\epsilon^{m+1})$$

on the time-scale 1.

3 The Poincaré Expansion Theorem

Theorem 9.2 (Poincaré Expansion Theorem) Consider the initial value problem

$$\dot{y} = F(t,y,\epsilon), \quad y(t_0) = \mu,$$

with $|t - t_0| \leq h, y \in D \subset R^n, 0 \leq \epsilon \leq \epsilon_0, 0 \leq \mu \leq \mu_0$. If $F(t,y,\epsilon)$ is continuous w.r.t. t, y and ϵ and can be expanded in a convergent power series w.r.t. y and ϵ for $||y|| \leq \rho, 0 \leq \epsilon \leq \epsilon_0$, then $y(t)$ can be expanded in a convergent power series w.r.t. ϵ and μ in a neighborhood of $\epsilon = \mu = 0$, convergent on the time-scale 1.