Introduction to Bifurcation Theory. In this chapter, we will cover the following materials:

- **Center Manifold Theory** allows us to reduce the dimension of a problem, you will most likely still be left with a nonlinear system.
- **Normal Form Theory** can be used to “simplify” the nonlinear system by (removing as much nonlinearity as possible. This involves nonlinear coordinate transformation.
- **Local Bifurcation Theory** uses the above techniques to determine when the system changes qualitatively as parameters are varied.

1 Center Manifold Theory

1.1 Existence

**Theorem 13.3 (Existence).** Consider

\[ \dot{x} = Ax + f(x) \]

where

1. \( x \in \mathbb{R}^n \) and \( A \) is a constant \( n \times n \) matrix; \( x = 0 \) is an isolated critical point; the vector function \( f(x) \) is \( C^k, k \geq 2 \), in a neighborhood of \( x = 0 \) and \( \lim_{||x|| \to 0} ||f(x)||/||x|| = 0 \);

2. the stable and unstable manifolds of equation

\[ \dot{y} = Ay \]

are \( E_s \) and \( E_u \), the space of eigenvectors corresponding with eigenvalues with zero real part is \( E_c \).

Then there exists a \( C^{k-1} \) invariant manifold \( W_c \), the center manifold, which is tangent to \( E_c \) near \( x = 0 \); if \( k = \infty \), then \( W_c \) is in general \( C^{m} \) with \( m \leq \infty \).
Example 13.6

\[
\begin{align*}
\dot{x} &= -x + y^2 \\
\dot{y} &= -y^3 + x^2
\end{align*}
\]

Example 13.7 \((W_c \text{ is not unique})\).

\[
\begin{align*}
\dot{x} &= x^2 \\
\dot{y} &= -y
\end{align*}
\]

1.2 Stability

Theorem 13.4 (Stability) Consider equation (13.19)

\[
\begin{align*}
\dot{x} &= Ax + f(x, y) \\
\dot{y} &= By + g(x, y)
\end{align*}
\]

where \(A\) has only eigenvalues with zero real part and \(B\) has only eigenvalues with negative real part; \(f\) and \(g\) have a Taylor expansion near \((0, 0)\). Then the flow in the center manifold is determined by the following equation (13.20)

\[
\dot{u} = Au + f(u, h(u)).
\]

where \(y = h(x)\) represents the center manifold of equation (13.19) near the isolated critical point \((0, 0)\). If the solution \(u = 0\) of the equation 13.20 is stable (unstable), then the solution near \((0, 0)\) of equation 13.19 is stable (unstable).
1.3 Approximation

The center manifold $h(x)$ can be approximated by substituting a Taylor expansion into the following PDE:

$$\frac{\partial h}{\partial x} (Ax + f(x, h)) - Bh - g(x, h) = 0.$$ 

Example 1.1.1

\begin{align*}
\dot{x} &= x^2 y - x^5 \\
\dot{y} &= -y + x^2
\end{align*}

Remark: The failure of the tangent space approximation.
Example 1.1.2

\[ \dot{x} = -xy - x^6 \]
\[ \dot{y} = -y + x^2 \]

Example 13.8

\[ \dot{x} = xy + ax^3 + bxy^2 \]
\[ \dot{y} = -y + cx^2 + dx^2y. \]

1.4 Center Manifolds Depending on Parameters

The Lorenz Equations.

\[ \dot{x} = \sigma(y - x), \]
\[ \dot{y} = \bar{\rho}x + x - y - xz, \]
\[ \dot{z} = -\beta z + xy. \]