# CDS 140b Final Project Introduction

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This handout is meant to introduce you to the final projects, which are due at the end of the term in place of a final exam. I will go over the steps in more detail as part of Thursday's lecture, February 5th.

## Introduction

The goal of the final project is to give you some familiarity with applying the methods you are learning in this class to some physical problems of practical interest, which have a phase space dimension of four. The main goal is to find periodic orbits around equilibrium points in these systems, along with the local and global dynamics associated with these orbits. The two example problems you can choose between come from the fields of atomic physics and planetary science, respectively:

- (Atomic physics) Dynamics of the Rydberg atom
- (Planetary science) Dynamics of a binary asteroid pair

The dynamical equations describing each problem are a function of only one parameter. The problems are described in more detail later.

## Questions to Address

Your assignment is to write a report in TEX(preferred) or by hand addressing some questions about these systems, showing your procedure and calculations. As part of the project, you will need to do some numerical computations using, for example, Matlab.

The questions you will need to address are summarized in the following steps. We hope that everyone can perform steps 1, 2, 3, and hopefully 4. Attempt step 5 if you can. A Matlab software package is available for steps 3, 4, and 5. The Matlab software was originally designed for another dynamical system and will be given as an example (which you will need to modify).

1. **Equilibrium points.** Find the equilibrium points for the system: their location and stability type. How do the equilibrium points change their location and stability as a function of the system parameter?

2. **Existence of periodic orbits and low order approximations.** The two systems have at least one equilibrium point of the type saddle $\times$ center. Linearize the vector field about one of these points. Do periodic orbits exist around this point in the linearized vector field? Can you determine their stability? Find analytical approximations to the periodic orbits.

3. **Higher order approximations using differential correction and continuation.** Analytical approximations of low order can give qualitative insight, but suppose one wants to compute a periodic orbit to any desired accuracy, or a periodic orbit with a large amplitude about the equilibrium point? One then needs to use differential correction, which Koon introduced in Lecture 2B. Differential correction involves using the analytical approximation as a first guess in an iterative numerical process to generate a higher order accurate periodic orbit. One computes the state transition matrix for a

solution of the vector field, which comes from solving the variational equations of the vector field. For an $n$-dimensional vector field $\dot{x} = f(x)$, the variational equations are an $n^2$-dimensional vector field. Thus, for $n = 4$, the variational equations are a 16-dimensional vector field. One can use Matlab software to numerically solve the equations, an example of which will be provided. To find periodic orbits of large amplitude, one uses a process of continuation. This means that one begins with small, differentially corrected (i.e., accurate) period orbits, and uses this as a seed for finding larger periodic orbits.

4. **Invariant manifolds of periodic orbits.** The eigenvalues of the state transition matrix of a differentially corrected periodic orbit give the stability of the orbit. We expect that for the systems you will be studying, two eigenvalues will be 1, while the other two are a self-reciprocal pair $(\lambda, 1/\lambda)$, where $\lambda > 1$. Corresponding to the eigenvalue $\lambda$ is the eigenvector in the unstable direction. Taking a small displacement from the periodic orbit in the unstable direction, one can use this as an initial condition to grow the unstable invariant manifold of the periodic orbit. This process is called globalization of the unstable invariant manifold and is performed by numerically integrating the initial condition under the original vector field, using a numerical integration software, e.g., a Matlab package. A similar process can be done to obtain the stable invariant manifold.

5. **Poincaré section.** To visualize the motion of the unstable manifold in the phase space, one can take a Poincaré section, i.e., a slice of the phase space which produces cross-sections of the unstable manifold of the periodic orbit.

## Introduction to the Example Problems

We now describe the two example problems. Both examples are time-independent Hamiltonian systems with two degree of freedom (phase space dimension of four). They both describe the motion of a particle in an external field, viewed in a rotating frame. The phase space coordinates are $(x, y, v_x, v_y)$, two position coordinates and two velocity coordinates. Furthermore, the equations of motion of both systems admit a first integral to the motion, known as the energy, which has the form:

$$E(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) + U(x, y),$$

where $U(x, y)$ is the effective potential in each system.

## Example Problem 1: Dynamics of the Rydberg Atom

The Rydberg atom that we study is concerned with the motion of an electron around the rest of the atom in crossed electric and magnetic fields. The electron is assumed to be in an excited energy level which is high enough such that its motion can be treated classically. The equations of motion for the electron can be written as

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{y} &= v_y, \\ \dot{v_x} &= -v_y - \frac{\partial U}{\partial x}, \\ \dot{v_y} &= v_x - \frac{\partial U}{\partial y}, \end{aligned} \tag{1}$$

where the effective potential is

$$U(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} - \varepsilon x.$$

This system has one free parameter, the scaled electric field parameter, $\varepsilon$, which typically has a value between 0.4 and 0.6.

## Example Problem 2: Dynamics of a Binary Asteroid Pair

Approximately 10% of observed near Earth asteroids are binary asteroid pairs. A simple model for a binary asteroid pair is used in this study; the planar, restricted, full two-body problem. The model describes the motion of a particle in the gravitational field of a massive elliptical asteroid of uniform density. The particle has no effect on the massive body. The equations of motion for the particle in a frame co-rotating with the asteroid are

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{y} &= v_y, \\ \dot{v_x} &= 2v_y - \frac{\partial U}{\partial x}, \\ \dot{v_y} &= -2v_x - \frac{\partial U}{\partial y}, \end{aligned} \tag{2}$$

where the effective potential is

$$U(x,y) = -\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}(x^2+y^2) - \frac{3C_{22}\left(x^2-y^2\right)}{\left(x^2+y^2\right)^{5/2}}.$$

This system has one free parameter, the gravitational field coefficient, $C_{22}$, which typically has a value between 0.1 and 0.4.