CALTECH
Control & Dynamical Systems

Final Project

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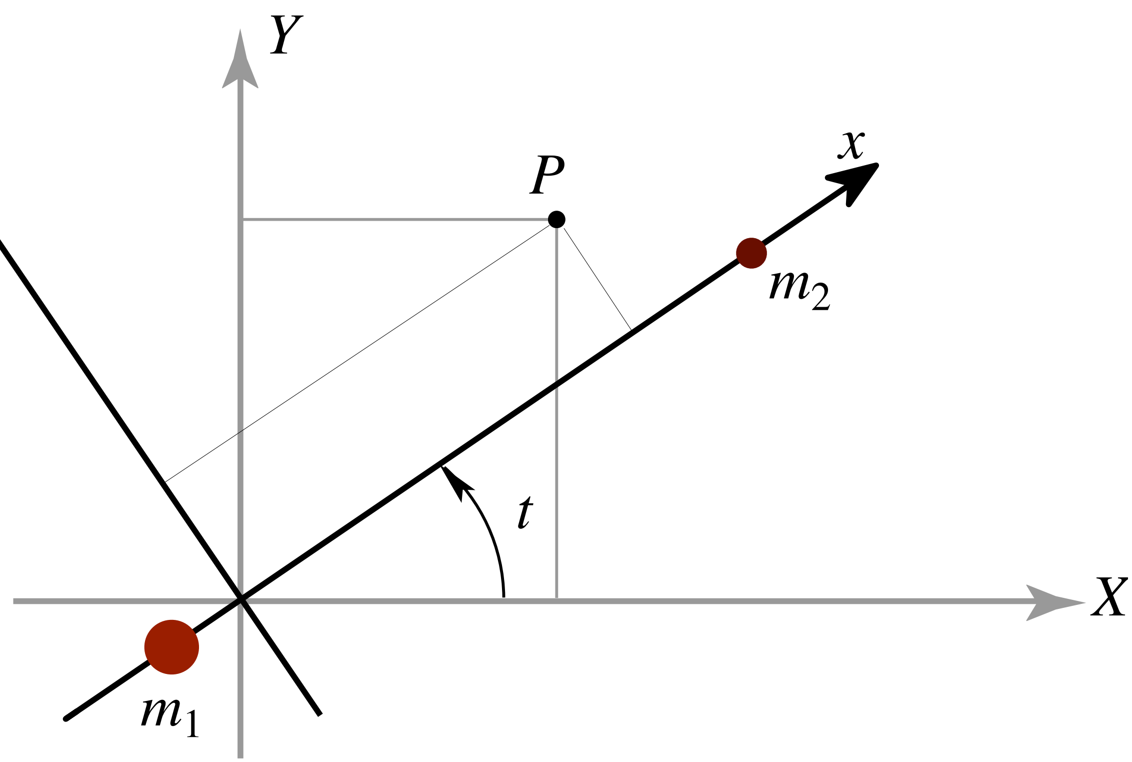
Final Project

- ***Issues to address in project***
  - Equilibrium points
  - Periodic orbits
    – low order analytical approximations
  - More accurate p.o.'s
    – Higher order numerical approximations of p.o.'s using differential correction and continuation
  - To be covered later
    – Invariant manifolds of p.o.'s
    – Poincaré section

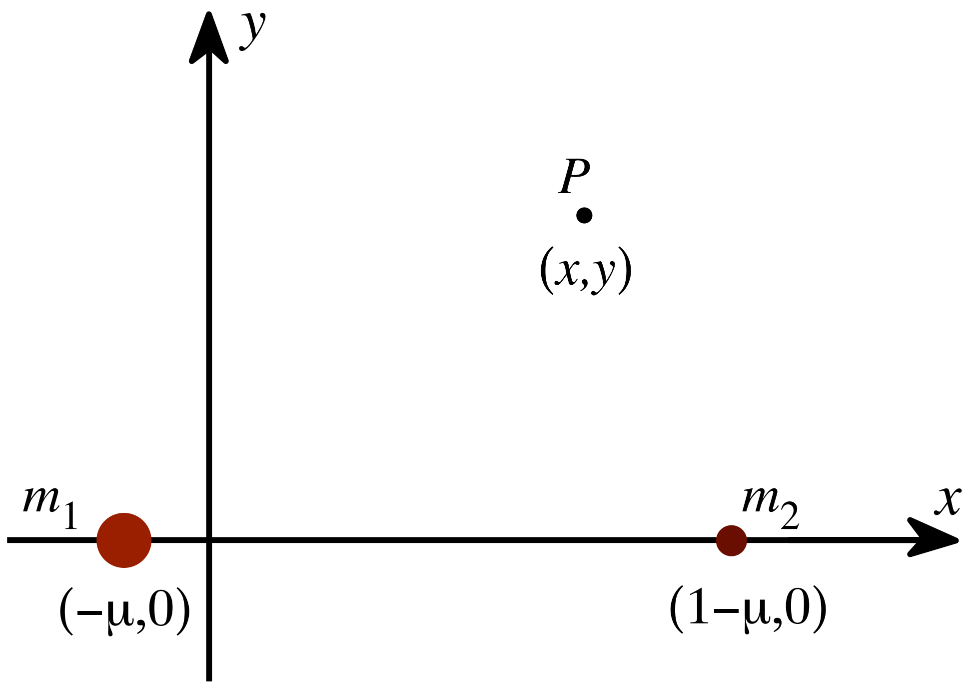
Example Problem

- **Planar, circular, restricted 3-body problem** (3BP)
  - From Chapter 2 of KLMR book (on class website)
  - P in field of two bodies, m_1 and m_2
  - x-y frame rotates w.r.t. X-Y inertial frame

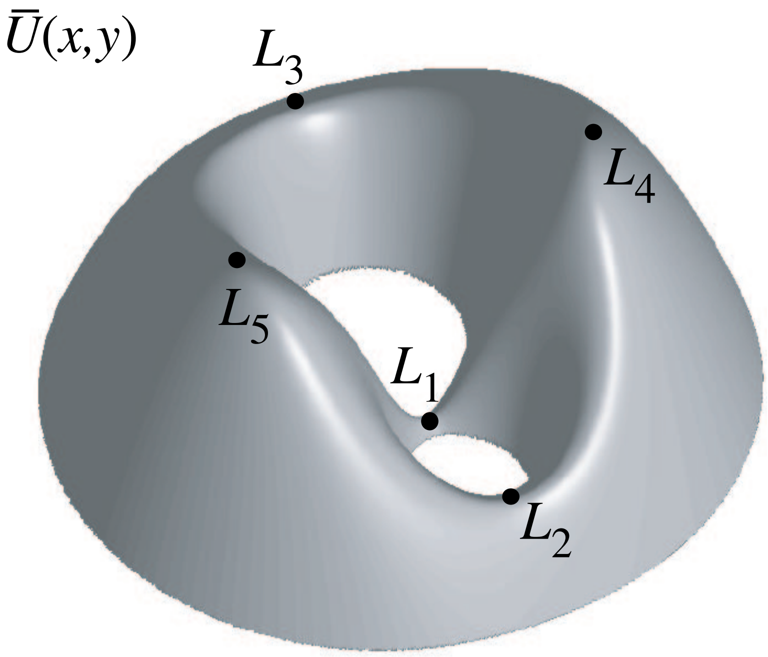


Example Problem

- Equations of motion describe P moving in an effective potential $\bar{U}(x.y)$ in a rotating frame



Position Space



Effective Potential

Example Problem

- Point in phase space: $q = (x\ y\ v_x\ v_y)^{\mathrm{T}} \in \mathbb{R}^4$
- Equations of motion, $\dot{q} = f(q)$, are

$$\begin{aligned}
\dot{x} &= v_x, \\
\dot{y} &= v_y, \\
\dot{v}_x &= 2v_y - \frac{\partial \bar{U}}{\partial x}, \\
\dot{v}_y &= -2v_x - \frac{\partial \bar{U}}{\partial y},
\end{aligned}$$

where

$$\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$$

where r_1 and r_2 are the distances of P from m_1 and m_2

Example Problem

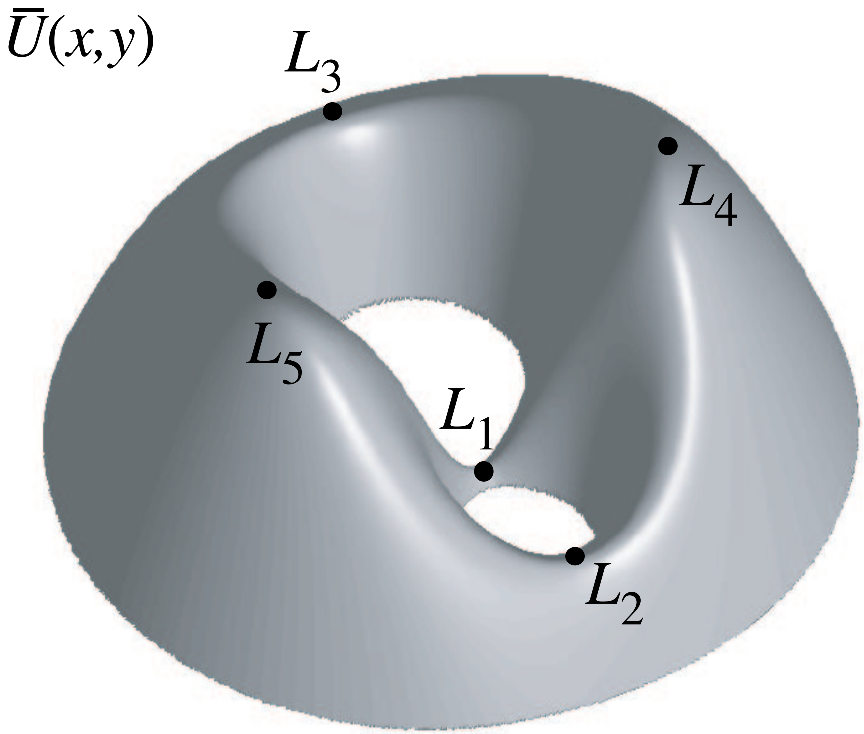
and the only parameter of the system is

$$\mu = \frac{m_2}{m_1 + m_2}$$

where $\mu \in (0, 0.5)$

Equilibrium points

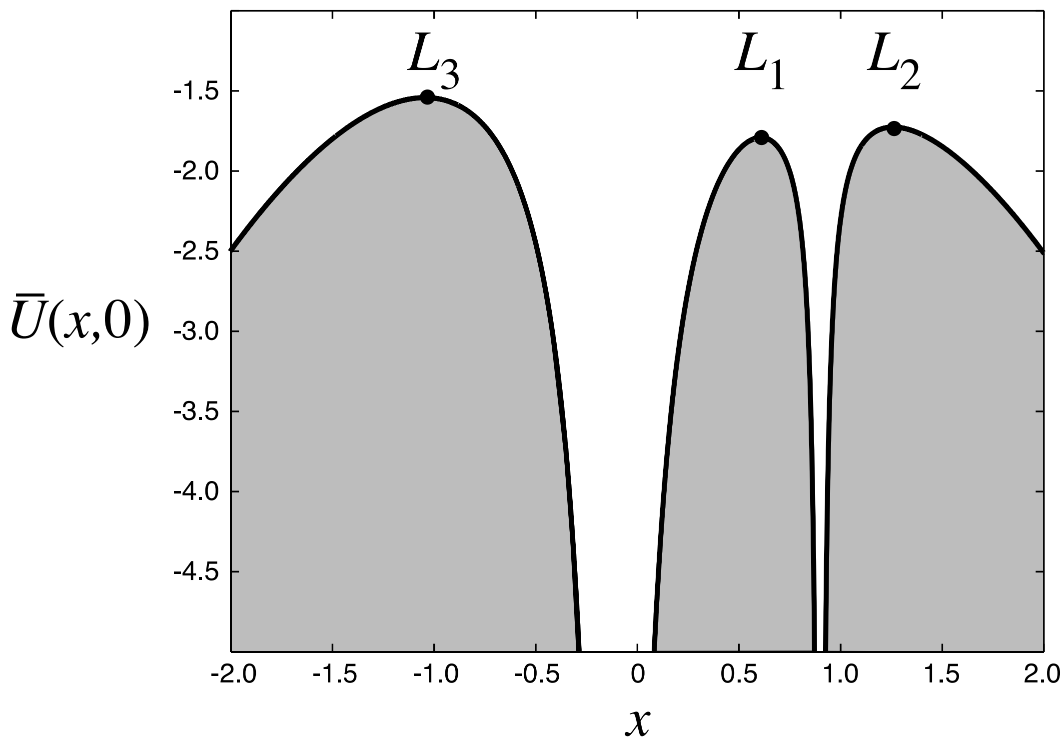
- Find $\bar{q} = (\bar{x}\ \bar{y}\ \bar{v}_x\ \bar{v}_y)^{\mathrm{T}}$ s.t. $\dot{\bar{q}} = f(\bar{q}) = 0$
- Have form $(\bar{x}, \bar{y}, 0, 0)$ where $(\bar{x}, \bar{y})$ are critical points of $\bar{U}(x, y)$, i.e., $\bar{U}_x = \bar{U}_y = 0$, where $U_a = \frac{\partial \bar{U}}{\partial a}$



Critical Points of $\bar{U}(x, y)$

Equilibrium points

- Consider x-axis solutions
- $\bar{U}_x = \bar{U}_y = 0 \Rightarrow$ polynomial in x
- depends on parameter μ



The graph of $\bar{U}(x,0)$ for $\mu = 0.1$

Equilibrium points

Phase space near equilibrium points

- Transform coordinates, placing $\bar{q}$ at origin,

$$q = \bar{q} + u$$

- Linearize vector field about $\bar{q}$

$$\dot{q} = \dot{\bar{q}} + \dot{u} = f(\bar{q}) + Df(\bar{q})u + \mathcal{O}(|u|^2).$$

- Since $\dot{\bar{q}} = f(\bar{q}) = 0$, we have

$$\dot{u} = Df(\bar{q})u + \mathcal{O}(|u|^2),$$

where $Df(\bar{q}) =$ a constant matrix.

Equilibrium points

- In 3BP, we have

$$A = Df(\bar{q}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{yx} & -U_{yy} & -2 & 0 \end{pmatrix}_{\bar{q}}.$$

- Eigenvalues of A tell us stability

Equilibrium points

- The x-axis solutions have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 2 \\ 0 & -b & -2 & 0 \end{pmatrix}$$

  where a and b are positive constants.

- Eigenvalues are $\pm\lambda$ and $\pm i\nu$.
- saddle $\times$ center geometry

$$\dim(E_s) = 1$$
$$\dim(E_u) = 1$$
$$\dim(E_c) = 2$$

Periodic orbits

Low order approximation methods

- Eigenvector method (Ch. 2 of KLMR)
- Naive method (Verhulst, Ch. 9)
- Poincaré-Lindstedt (Verhulst, Ch. 9 & Lecture 2B)

Periodic orbits

Eigenvector method for 3BP

- Eigenvalues of the linear system are $\pm\lambda$ and $\pm i\nu$ with corresponding eigenvectors u_1, u_2, w_1, w_2.
- Thus, the general (real) solution has the form

$$
\begin{aligned}
u(t) &= (x(t)\ y(t)\ v_x(t)\ v_y(t))^{\mathrm{T}}, \\
&= \alpha_1 e^{\lambda t} u_1 + \alpha_2 e^{-\lambda t} u_2 + 2\mathrm{Re}(\beta e^{i\nu t} w_1),
\end{aligned}
$$

  where α_1, α_2 are real and $\beta = \beta_1 + i\beta_2$ is complex.
- $\alpha_1 = \alpha_2 = 0 \Rightarrow$ a *periodic orbit* of period $T = \frac{2\pi}{\nu}$.
- A theorem of Moser [1958] guarantees the existence of the p.o. in the full nonlinear equations.

More accurate p.o.'s

- Approximation methods above may not give a p.o., $x_{\rm po}(t)$, of a desired accuracy
- The p.o. may be unstable
- We want $\bar{x}_{\rm po}(0)$ s.t.

$$|\bar{x}_{\rm po}(T) - \bar{x}_{\rm po}(0)| < \epsilon$$

  for specified ϵ

More accurate p.o.'s

- ***How to get accurate, high amplitude p.o.'s?***

- high order analytic expansion (e.g., Poincaré-Lindstedt)
- normal form theory (to high order)
- numerical differential correction and continuation

- Lecture 2B discussed diff. corr.; we review here
- See also Chs. 6 & 7 of KLMR

More accurate p.o.'s

Differential correction

- Given $\bar{x}(t)$ going from $\bar{x}_0$ to $\bar{x}_1$ under

$$\dot{x} = f(x),$$

  wiggle $\bar{x}_0$ by $\delta\bar{x}_0$ so trajectory will end at x_d, near $\bar{x}_1$.
- Need sensitivity of $\delta\bar{x}_1$ w.r.t. $\delta\bar{x}_0$.
- Linear approx., **state transition matrix**.

More accurate p.o.'s

- Let trajectories with $\bar{x}(t_0) = \bar{x}_0$ be denoted by $\phi(t, t_0; \bar{x}_0)$.
- perturbed initial vector $\bar{x}_0 + \delta\bar{x}_0$ evolves as

$$\delta\bar{x}(t) = \phi(t, t_0; \bar{x}_0 + \delta\bar{x}_0) - \phi(t, \bar{x}_0)$$

w.r.t. reference trajectory $\bar{x}(t)$.

More accurate p.o.'s

- Measuring the distance at time t_1 gives

$$\delta\bar{x}(t_1) = \phi(t_1, t_0; \bar{x}_0 + \delta\bar{x}_0) - \phi(t_1, t_0; \bar{x}_0).$$

- Taylor expansion yields

$$\delta\bar{x}(t_1) = \frac{\partial\phi(t_1, t_0; \bar{x}_0)}{\partial x_0}\delta\bar{x}_0 + \mathcal{O}(|\delta\bar{x}_0|^2)$$

- The matrix $\frac{\partial\phi(t_1,t_0;\bar{x}_0)}{\partial x_0}$ which satisfies tha above relation to first order is called the **state transition matrix**.
- Abbreviated as $\Phi(t_1, t_0)$, this matrix, given by

$$\delta\bar{x}(t_1) = \Phi(t_1, t_0)\delta\bar{x}_0,$$

plays a key role in differential correction.

More accurate p.o.'s

- Suppose,

$$\bar{x}(t_1) = \phi(t_1, t_0; \bar{x}_0) = \bar{x}_1 = x_d - \delta\bar{x}_1,$$

  is slightly off ($|\delta\bar{x}_1| > \epsilon$) and we need to correct it.

- Since

$$\begin{aligned}
\phi(t_1, t_0; \bar{x}_0 + \delta\bar{x}_0) &\approx \phi(t_1, t_0; \bar{x}_0) + \frac{\partial\phi(t_1, t_0; \bar{x}_0)}{\partial x_0}\delta\bar{x}_0 \\
&\approx \phi(t_1, t_0; \bar{x}_0) + \Phi(t_1, t_0)\delta\bar{x}_0 \\
&\approx \bar{x}_1 + \delta\bar{x}_1 \\
&\approx x_d,
\end{aligned}$$

  $\Rightarrow$ changing $\bar{x}_0$ by $\delta\bar{x}_0 = \Phi(t_1, t_0)^{-1}\delta\bar{x}_1$ works to first order.

More accurate p.o.'s

- By iteration, the process produces convergence:

$$|\phi(t_1, t_0; \bar{x}_0 + \Delta\bar{x}_0) - x_d| < \varepsilon$$

where $\Delta\bar{x}_0$ is the accumulation of corrections $\delta\bar{x}_0$ which yields x_d within the desired tolerance ε.

More accurate p.o.'s

■ *Computation of* $\Phi(t_1, t_0)$

- Since ϕ satisfies

$$\frac{d\phi(t;\bar{x}_0)}{dt} = f(\phi(t,\bar{x}_0)),$$

with $\phi(t_0;\bar{x}_0) = \bar{x}_0$, diff. w.r.t. x_0 yields

$$\frac{d}{dt}\frac{\partial\phi(t;\bar{x}_0)}{\partial x_0} = Df(\phi)\frac{\partial\phi(t;\bar{x}_0)}{\partial x_0},$$

where $\frac{\partial\phi(t_0;\bar{x}_0)}{\partial x_0} = I$.

- Hence, $\Phi(t, t_0)$ solves the following initial value problem

$$\dot{\Phi}(t,t_0) = Df(\bar{x}(t))\Phi(t,t_0), \quad \Phi(t_0,t_0) = I$$

More accurate p.o.'s

- $Df(\bar{x}(t))$ obtained numerically in general.
- Thus, $\Phi(t,0)$ along a reference orbit $\bar{x}(t)$ is computed by numerically integrating $n+n^2$ ODEs:

$$\begin{aligned} \dot{\bar{x}}(t) &= f(\bar{x}(t)), \\ \dot{\Phi}(t,0) &= Df(\bar{x}(t))\Phi(t,0), \end{aligned}$$

with initial conditions:

$$\begin{aligned} \bar{x}(0) &= \bar{x}_0, \\ \Phi(0,0) &= I, \end{aligned}$$

where we've set $t_0 = 0$.

More accurate p.o.'s

- In 3BP, we have

$$Df(\bar{x}(t)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{yx} & -U_{yy} & -2 & 0 \end{pmatrix}_{\bar{x}(t)} .$$

More accurate p.o.'s

- We seek periodic orbits which are symmetric w.r.t. the x-axis ($y = 0$), noting that $y \mapsto -y, t \mapsto -t$ leaves equations of motion unchanged.

  – Gives mirror image solution $\bar{x}(-t)$ for each $\bar{x}(t)$

- Intersect x-axis perpendicularly,

$$\bar{x}(0) = (x_0 \ 0 \ 0 \ v_{y0})^{\mathrm{T}}$$

- We get this first guess from approx. methods.

More accurate p.o.'s

- ODEs integrated until next x-axis crossing
  - integrate until $y(t)$ changes sign
  - then change time step until, e.g., $|y(t)| < 10^{-11}$,
  - at crossing, $t \equiv t_1$, $y_1 \equiv y(t_1)$
- We have $\bar{x}(t_1)$, so compute $\Phi(t_1, 0)$ as well
- For p.o., desired final state has form

$$\bar{x}(t_1) = (x_1 \; 0 \; 0 \; v_{y1})^{\mathrm{T}}$$

  where $t_1 = T/2$
- actual value for v_{x1} may not be 0
- we want $|v_{x1}| < \varepsilon$, e.g., $\varepsilon = 10^{-8}$.

More accurate p.o.'s

- $\Phi(t_1, 0)$ can be used to adjust the initial values to obtain a p.o. as

$$\delta \bar{x}_1 \approx \Phi(t_1, 0)\delta \bar{x}_0 + \dot{\bar{x}}_1 \delta t_1$$

- Assume $|v_{x1}| > \varepsilon$ and we hold x_0 fixed
- Correction to v_{y0} can be calculated from

$$\delta v_{x1} \approx \Phi_{34}\delta v_{y0} + \dot{v}_{x1}\delta t_1$$
$$0 = \delta y_1 \approx \Phi_{24}\delta v_{y0} + v_{y1}\delta t_1$$

  where Φ_{ij} is an element of matrix $\Phi(t_1, 0)$.
- Here, $\delta v_{x1} = v_{x1}$ since we want $v_{x1} = 0$

More accurate p.o.'s

- Hence,

$$\delta v_{y0} \approx \left(\Phi_{34} - \frac{1}{v_{y1}} \Phi_{24} \right)^{-1} v_{x1}$$

  can be used to cancel out v_{x1} if we let

$$v_{y0} \mapsto v_{y0} - \delta v_{y0}$$

- This process converges to $|v_{x1}| < 10^{-8}$ within a few iterations

More accurate p.o.'s

Numerical Continuation

- Suppose we find two small nearby p.o. initial conditions using diff. cor.

$$\bar{x}_0^{(1)}, \bar{x}_0^{(2)}$$

- We can generate a family of p.o.'s with increasing amplitude around an equil. point in the following way (using 3BP's L_1 as an example).
- Let

$$\begin{aligned} \Delta &= \bar{x}_0^{(2)} - \bar{x}_0^{(1)}, \\ &= (\Delta x_0 \; 0 \; 0 \; \Delta v_{y0})^{\mathrm{T}} \end{aligned}$$

More accurate p.o.'s

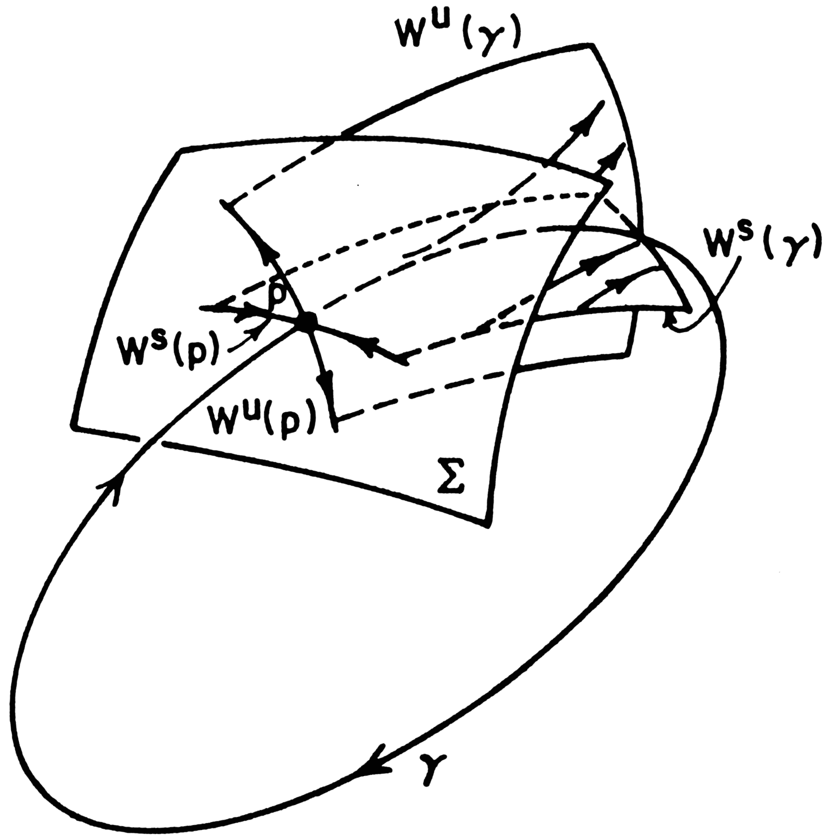
- Then extrapolate to an initial guess for $\bar{x}_0^{(3)}$ via

$$\begin{aligned}\bar{x}_0^{(3)} &= \bar{x}_0^{(2)} + \Delta, \\ &= \left((x_0^{(2)} + \Delta x_0) \; 0 \; 0 \; (v_{y0}^{(2)} + \Delta v_{y0}) \right)^{\mathrm{T}} \\ &= \left(x_0^{(3)} \; 0 \; 0 \; v_{y0}^{(3)} \right)^{\mathrm{T}}\end{aligned}$$

- Keeping $x_0^{(3)}$ fixed, we can use differential correction to compute an accurate solution $\bar{x}_0^{(3)}$ and repeat the process until we have a family of solutions.

Invariant manifolds of p.o.'s

- The **monodromy matrix** $\Phi(T, 0)$ has an unstable and stable eigenvector. We can numerically integrate this linear approximation to the unstable (or stable) direction to obtain the unstable (or stable) manifold.



Poincaré Sections

- This set of solutions approximating the unstable manifold can be numerically integrated until some stopping condition is reached (e.g., $x_j = \mathrm{constant}$).

References

- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2004] ***Dynamical systems, the three-body problem, and space mission design,*** preprint.
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics, ***Chaos*** 10(2), 427–469.