The Method of Averaging

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1 Introduction

Remarks: The method leads generally to asymtotic series as opposed to convergent series. It is not restriced to periodic solutions.

Averaging Method. Put the equation

$$\ddot{x} + x = \epsilon f(x, \dot{x})$$

into Lagrange stardard form and do the averaging.

Example 11.1

$$\ddot{x} + x = \epsilon(-\dot{x} + x^2).$$

2 The Lagrange standard form

Unperturbed Equation is Linear.

$$\dot{x} = A(t)x + \epsilon g(t, x), \qquad x(0) = x_0.$$

3 Avaraging in the Periodic Case

Asymptotic Validity of Averaging Method. Consider equation (11.17)

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \qquad x(0) = x_0.$$

We assume that $f(t, x)$ is T-periodic in t and we introduce the average

$$f^0(y) = \frac{1}{T} \int_0^T f(t, y) dt.$$

Consider now equation (11.18)

$$\dot{y} = \epsilon f^0(y), \qquad y(0) = x_0.$$

Theorem 11.1 Consider the initial value problem 11.7 and 11.8 with $x, y, x_0 \in D \subset R^n, t \geq 0$. Suppose that

1. f, g and $\partial f / \partial x$ are defined, continuous and bounded by a constant M in $[0, \infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. $f(t, x)$ is T-periodic in t with average $f^0(x)$ where T is a constant independent of ϵ;
4. $y(t)$ is contained in the interior of D.

Then we have $x(t) - y(t) = O(\epsilon)$ on the time-scale $1/\epsilon$.

Remark on Example 11.1: The estimates are not valid if we start near the saddle point $x = 1/\epsilon, \dot{x} = 0$.

Example 11.3 Consider

$$\ddot{x} + x = \epsilon f(x, \dot{x})$$

and the van der Pol equation

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}.$$

4 Averaging in the General Case

Theorem 11.2 Consider the initial value problem

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \qquad x(0) = x_0.$$

with $x, x_0 \in D \subset R^n, t \geq 0$. Assume that

1. f, g and $\partial f / \partial x$ are defined, continuous and bounded by a constant in $[0, \infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. $f(t, x) = \sum_{i=1}^{N} f_i(t, x)$ with $f_i(t, x)$ being T_i-periodic in t where T_i constants independent of ϵ;
4. $y(t)$ is ths solution of the initial value problem

   $$\dot{y} = \epsilon \sum_{i=1}^{N} \frac{1}{T_i} \int_0^{T_i} f_i(t, y) dt, \qquad y(0) = x_0.$$

   and $y(t)$ is contained in the interior of D.

Then we have $x(t) - y(t) = O(\epsilon)$ on the time-scale $1/\epsilon$.

Theorem 11.3 Consider the initial value problem

$$\dot{x} = \epsilon f(t,x) + \epsilon^2 g(t,x,\epsilon), \qquad x(0) = x_0.$$

with $x, x_0 \in D \subset R^n, t \geq 0$. Assume that

1. f, g and $\partial f/\partial x$ are defined, continuous and bounded by a constant in $[0,\infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. the average $f^0(x)$ of $f(t,x)$ exists where

$$f^0(x) = \text{lim}_{t\to\infty} \frac{1}{T} \int_0^T f(t,x)dt;$$

4. $y(t)$ is ths solution of the initial value problem

$$\dot{y} = \epsilon f^0(y), \qquad y(0) = x_0.$$

   and $y(t)$ is contained in the interior of D.

Then we have $x(t) - y(t) = O(\delta(\epsilon))$ on the time-scale $1/\epsilon$ with

$$\delta(\epsilon) = \text{sup}_{x\in D}\text{sup}_{0\leq \epsilon t\leq C} || \int_0^t [f(s,x) - f^0(x)]ds||.$$

5 Adiabatic Invariants

Consider

$$\dot{x} = \epsilon f(t, \epsilon t, x), \qquad x(0) = x_0.$$

Introduce $\tau = \epsilon t$, we have

$$\begin{aligned} \dot{x} &= \epsilon f(t,\tau,x), \qquad x(0) = x_0 \\ \dot{\tau} &= \epsilon, \qquad\qquad\quad \tau(0) = 0. \end{aligned}$$

Suppose we can average the system above over t with averaged equations

$$\begin{aligned} \dot{y} &= \epsilon f^0(\tau,y)), \qquad y(0) = x_0 \\ \dot{\tau} &= \epsilon, \qquad\qquad\quad \tau(0) = 0. \end{aligned}$$

If we can solve this system, then by replacing $\tau = \epsilon t$, we obtain an approximation of $x(t)$.

Example 11.6: Linear oscillator with slowing varying frequency.

$$\ddot{x} + \omega^2(\epsilon t)x = 0.$$

Remark: Such a quantity which has been conserved asymptotically while the coefficients are varying slowly with time is called an *adiabatic invariant*.

6 Periodic Solutions

Theorem 11.5 Consider equation (11.48)

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon)$$

with $x \in D \subset R^n, t \geq 0$. Suppose that

1. $f, g, \partial f/\partial x, \partial^2 f/\partial x^2$ and $\partial g/\partial x$ are defined, continuous and bounded by a constant M in $[0, \infty) \times D, 0 \leq \epsilon \leq \epsilon_0$;
2. f and g are T-periodic in t.

If p is critical point of the averaged equation

$$\dot{y} = \epsilon^0(y),$$

with $|\partial f^0 f(y)/\partial y|_{y=p} \neq 0$, then there exists a T-periodic solution $\phi(t, \epsilon)$ of equation (11.48) which is close to p such that

$$\lim_{\epsilon \leftarrow 0} \phi(t, \epsilon) = p.$$

Theorem 11.6 Consider equation (11.48) and suppose that the conditions of theorem 11.5 have been satisfied. If the eigenvalues of the critical point $y = p$ of the averaged equation have all negative real parts, the corresponding periodic solution $\phi(t, \epsilon)$ of equation (11.48) is aymptotically stable for ϵ sufficiently small. If one of the eigenvalues has positive real part, $\phi(t, \epsilon)$ is unstable.

Example 11.9 (autonomous equations) Van der Pol Equation.

Example 11.10: Forced Duffing Equation.