Lecture 6

11. Adaptive Control in the Presence of Bounded Disturbances

Consider MIMO systems in the form,

$$\begin{aligned} \dot{x} &= A_{ref}\, x + B\,\Lambda\left(u + \Theta^T\, \Phi\left(x\right)\right) + B_{ref}\, y_{cmd} + \xi\left(t\right) \\ y &= C_{ref}\, x \end{aligned} \tag{11.1}$$

operating in the presence of a bounded time-dependent disturbance $\xi\left(t\right) \in R^n$. All the assumptions and notations from the previous section apply here, that is we assume that $\left(A_{ref},\, B,\, B_{ref},\, C_{ref}\right)$ are known, with A_{ref} being Hurwitz. The system matched uncertainties are represented by a diagonal positive definite matrix $\Lambda \in R^{m\times m}$ and a constant matrix $\Theta \in R^{N\times m}$. In addition, we assume that,

$$\left\|\xi\left(t\right)\right\| \le \xi_{\max} \tag{11.2}$$

and that the disturbance upper bound $\xi_{\max} \ge 0$ is *known* and *constant*. The *control goal* is *bounded tracking* of the reference model dynamics,

$$\begin{aligned} \dot{x}_{ref} &= A_{ref}\, x_{ref} + B_{ref}\, y_{cmd} \\ y_{ref} &= C_{ref}\, x_{ref} \end{aligned} \tag{11.3}$$

driven by a bounded time-dependent command $y_{cmd} \in R^m$.

Based on (11.1), the control input is selected as:

$$u = -\hat{\Theta}^T\, \Phi\left(x\right) \tag{11.4}$$

where $\hat{\Theta} \in R^{N\times m}$ is the matrix of adaptive parameters. Substituting (11.4) into (11.1), gives,

$$\dot{x} = A_{ref}\, x - B\,\Lambda\,\Delta\Theta^T\, \Phi\left(x\right) + B_{ref}\, y_{cmd} + \xi\left(t\right) \tag{11.5}$$

where,

$$\Delta\Theta = \hat{\Theta} - \Theta \tag{11.6}$$

is the matrix of estimation errors. Let,

$$e = x - x_{ref} \tag{11.7}$$

be the tracking error. Subtracting the reference model dynamics (11.3) from that of the system (11.1), yields the *tracking error dynamics*:

$$\dot{e} = A_{ref}\, e - B\,\Lambda\,\Delta\Theta^T\, \Phi\left(x\right) + \xi\left(t\right) \tag{11.8}$$

The Lyapunov function candidate is selected as in the previous section, that is:

$$V\left(e,\, \Delta\Theta\right) = e^T\, P\, e + \mathrm{trace}\left(\Delta\Theta^T\, \Gamma_\Theta^{-1}\, \Delta\Theta\,\Lambda\right) \tag{11.9}$$

where $\Gamma_\Theta = \Gamma_\Theta^T > 0$ denotes constant rates of adaptation, and $P = P^T > 0$ is the unique symmetric positive definite solution of the algebraic Lyapunov equation,

$$A_{ref}^T P + P A_{ref} = -Q \tag{11.10}$$

with $Q = Q^T > 0$. Time-differentiating V, along the trajectories of (11.8), gives:

$$\dot{V} = -e^T Q e - 2 e^T P B \Lambda \Delta\Theta^T \Phi(x) + 2 e^T P \xi(t) + 2\,\mathrm{trace}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} \Lambda\right) \quad (11.11)$$

Applying the trace identity,

$$a^T b = trace\left(b a^T\right) \quad (11.12)$$

yields:

$$\dot{V} = -e^T Q e + 2\,\mathrm{trace}\left(\Delta\Theta^T \left\{\Gamma_\Theta^{-1} \dot{\hat{\Theta}} - \Phi e^T P B\right\} \Lambda\right) + 2 e^T P \xi(t) \quad (11.13)$$

Suppose that we use the same adaptive laws as before, that is:

$$\dot{\hat{\Theta}} = \Gamma_\Theta \Phi(x)\, e^T P B \quad (11.14)$$

Then,

$$\dot{V} = -e^T Q e + 2 e^T P \xi(t) \le -\lambda_{\min}(Q)\|e\|^2 + 2\|e\|\, \lambda_{\max}(P)\xi_{\max} \quad (11.15)$$

and, consequently $\dot{V} < 0$ outside of the set,

$$E_0 = \left\{(e,\, \Delta\Theta):\quad \|e\| \le \frac{2\lambda_{\max}(P)\xi_{\max}}{\lambda_{\min}(Q)} = e_0\right\} \quad (11.16)$$

Hence, trajectories $\left(e(t),\, \Delta\Theta(t)\right)$, of the error dynamics (11.8) coupled with the adaptive law (11.14), enter the set E in finite time and stay there for all future times. However, the set E is *not* compact in the $(e,\, \Delta\Theta)$ – space. In fact, it is unbounded since $\Delta\Theta$ is not restricted. Inside the set, $\dot{V}$ can become positive and, as a consequence, the parameter errors $\Delta\Theta$ can grow unbounded, even though the tracking error norm remains less then e_0 at all times. This is the "parameter drift" phenomenon. It is caused by the disturbance term $\xi(t)$. It shows that the adaptive laws (11.14) are *not* robust to bounded disturbances, no matter how small the latter are.

Dead-Zone Modification

In order to enforce robustness, we consider adaptive laws with the *Dead-zone modification*:

$$\dot{\hat{\Theta}} = \begin{cases} \Gamma_\Theta \Phi(x)\, e^T P B, & \text{if } \|e\| > e_0 \\ 0_{N\times m}, & \text{if } \|e\| \le e_0 \end{cases} \quad (11.17)$$

Proposed by B.B. Peterson and K.S. Narendra in "Bounded error adaptive control," IEEE Transactions on Automatic Control, v. 27, 1161-1168, 1982, the dead-zone modification stops the adaptation process when the norm of the tracking error becomes smaller than the prescribed value e_0. This assures ultimate uniform boundedness (UUB) of $\Delta\Theta$, (in addition to UUB of e). We are going to formally prove this claim.

Suppose that $\|e\| > e_0$ then the adaptive law is defined by (11.14), and it results in the upper bound (11.15). Consequently, $e(t)$ enters E_0 in finite time T: $\|e(t+T)\| \le e_0$, for all $t \ge 0$. From that time forward, the adaptive parameter dynamics are frozen, that is $\dot{\hat{\Theta}}(t+T) = 0_{N\times m}$. Therefore, trajectories remain in the compact set,

$$\left(e(t),\ \Delta\Theta(t)\right) \in E_0 \times \Delta\Theta(T) \tag{11.18}$$

for all $t \geq 0$. This proves UUB of all signals in the corresponding closed-loop system.

Remark 11.1

In (11.16), the tracking error bound e_0 depends on the ratio $\dfrac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}$. Its maximum is achieved for $Q = I_{n\times n}$. Thus, with the dead-zone modification, the worst achievable tracking error value is:

$$e_{\max} = 2\,\lambda_{\max}(P)\,\xi_{\max} \tag{11.19}$$

Moreover, even when the disturbance vanishes, with the dead-zone modification being active, asymptotic stability of the tracking error cannot be recovered.

The dead-zone modification is *not* Lipschitz, and as such may cause chattering (high frequency oscillations) and other undesirable effects when the tracking error is at or near the dead-zone boundary. A smooth version of the dead-zone modification was proposed by J.-J. E. Slotine and J.A. Coetsee in "Adaptive sliding controller synthesis for nonlinear systems," International Journal of Control, 1986. Motivated by this idea, we choose a constant $0 < \delta < 1$, and consider a Lipchitz-continuous *modulation function* in the form,

$$\mu(\|e\|) = \max\left(0,\ \min\left(1,\ \frac{\|e\| - \delta\, e_0}{(1-\delta)\, e_0}\right)\right) \tag{11.20}$$

This function is shown in the figure below.

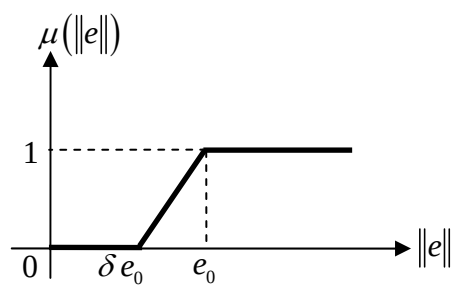

Figure 11.1. Lipschitz-Continuous Modulation Function.

The adaptive laws with continuous dead-zone modification are defined as,

$$\dot{\hat{\Theta}} = \Gamma_\Theta\, \Phi(x)\ \mu(\|e\|)\ e^T\, P\, B \tag{11.21}$$

With these laws of adaptation, one can use Lyapunov-based arguments to prove bounded tracking and UUB of all signals.

Sigma-Modification

The Dead-zone modification assumed prior knowledge of an upper bound $\xi_{\max}$ for the system disturbance $\xi(t)$. The σ*–modification* scheme, proposed by P. Ioannou and P. Kokotovic in "Adaptive systems with reduced models," New York: Springer-Verlag,

1983, does not require any prior information about bounds for the system disturbance. The adaptive law with $\sigma-$ modification is defined as:

$$\dot{\hat{\Theta}} = \Gamma_{\Theta}\left(\Phi(x)\, e^T\, P\, B - \sigma\,\hat{\Theta}\right) \tag{11.22}$$

Basically, this modification adds damping to the ideal adaptive law (11.14). To prove UUB of all signals, consider again the Lyapunov function candidate (11.9). Its time derivative along the trajectories of the tracking error dynamics (11.8) becomes:

$$\begin{aligned}
\dot{V} &= -e^T\, Q\, e + 2\,\mathrm{trace}\left(\Delta\Theta^T \left\{\Gamma_{\Theta}^{-1}\,\dot{\hat{\Theta}} - \Phi\, e^T\, P\, B\right\}\Lambda\right) + 2\, e^T\, P\, \xi(t) \\
&= -e^T\, Q\, e - 2\,\sigma\,\mathrm{trace}\left(\Delta\Theta^T \underbrace{\hat{\Theta}}_{\Theta+\Delta\Theta}\,\Lambda\right) + 2\, e^T\, P\, \xi(t) \\
&= -e^T\, Q\, e - 2\,\sigma\,\mathrm{trace}\left(\Delta\Theta^T\,\Delta\Theta\,\Lambda\right) - 2\,\sigma\,\mathrm{trace}\left(\Delta\Theta^T\,\Theta\,\Lambda\right) + 2\, e^T\, P\, \xi(t)
\end{aligned} \tag{11.23}$$

By definition:

$$\mathrm{trace}\left(\Delta\Theta^T\,\Delta\Theta\,\Lambda\right) = \sum_{i=1}^{N}\sum_{j=1}^{m} \Delta\Theta_{ij}^2\,\Lambda_{ii} \geq \left\|\Delta\Theta\right\|_F^2\,\Lambda_{\min} \tag{11.24}$$

where $\left\|\Delta\Theta\right\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}\Delta\Theta_{ij}^2$ is the Frobenius norm of $\Delta\Theta$, and $\Lambda_{\min}$ is the minimum diagonal element of Λ. Moreover, using the Schwarz inequality, gives:

$$\left|\mathrm{trace}\left(\Delta\Theta^T\,\Theta\,\Lambda\right)\right| \leq \left\|\Delta\Theta^T\,\Theta\right\|_F \left\|\Lambda\right\|_F \leq \left\|\Delta\Theta\right\|_F \left\|\Theta\right\|_F \left\|\Lambda\right\|_F \tag{11.25}$$

Substituting (11.24) and (11.25) into (11.23), results in:

$$\begin{aligned}
\dot{V} &\leq -\lambda_{\min}(Q)\left\|e\right\|^2 + 2\left\|e\right\|\lambda_{\max}(P)\,\xi_{\max} - 2\,\sigma\left\|\Delta\Theta\right\|_F^2\,\Lambda_{\min} + 2\,\sigma\left\|\Delta\Theta\right\|_F \left\|\Theta\right\|_F \left\|\Lambda\right\|_F \\
&= -\lambda_{\min}(Q)\left\|e\right\|\left(\left\|e\right\| - 2\frac{\lambda_{\max}(P)\,\xi_{\max}}{\lambda_{\min}(Q)}\right) - 2\,\sigma\,\Lambda_{\min}\left\|\Delta\Theta\right\|_F\left(\left\|\Delta\Theta\right\|_F - \frac{\left\|\Theta\right\|_F\left\|\Lambda\right\|_F}{\Lambda_{\min}}\right)
\end{aligned} \tag{11.26}$$

Hence, $\dot{V} < 0$ outside of the compact set,

$$\Omega = \left\{\left(e,\,\Delta\Theta\right) \in R^n \times R^{N\times m}:\quad \left\|e\right\| \leq 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\,\xi_{\max} \wedge \left\|\Delta\Theta\right\|_F \leq \frac{\left\|\Theta\right\|_F\left\|\Lambda\right\|_F}{\Lambda_{\min}}\right\} \tag{11.27}$$

This proves UUB of all signals in the closed-loop dynamics. In particular, (11.27) proves UUB tracking of the external command $y_{cmd}(t)$ by the system output $y(t)$, while the system operates in the presence of parametric uncertainties $(\Lambda,\,\Theta)$ and nonparametric bounded time-varying disturbances $\xi(t)$.

E-Modification

The drawback of the $\sigma-$ modification is that when the tracking error becomes small the adaptive parameters have a tendency to go back to the origin, that is they "unlearn" the gain values that caused the tracking error to become small. In order to overcome this undesirable effect, K.S. Narendra and A.M. Annaswamy developed the *e* – modification, [K.S. Narendra, A.M. Annaswamy, "A new adaptive law for robust adaptive control without persistency of excitation," *IEEE Transactions on Automatic Control*, 32:134-145, Feb. 1987].

The e – modification replaces the damping gain σ in (11.22) with a term proportional to $\left\| e^T P B \right\|$. The rational for using such a term is that it tends to 0 with the output error. With this modification, the adaptive laws inherit error-dependent damping effects. The adaptive laws with e – modification are chosen as:

$$\dot{\hat{\Theta}} = \Gamma_\Theta \left(\Phi(x)\, e^T P B - \left\| e^T P B \right\| \hat{\Theta} \right) \tag{11.28}$$

As seen from (11.28), the e – modification adds variable (tracking error dependent) damping to the adaptive laws dynamics. Using these laws, we now compute the time-derivative of the Lyapunov function candidate (11.9), along the trajectories of the tracking error dynamics (11.8). From (11.13), it follows:

$$\begin{aligned}
\dot{V} &= -e^T Q e + 2\,\mathrm{trace}\left(\Delta\Theta^T \left\{ \Gamma_\Theta^{-1} \dot{\hat{\Theta}} - \Phi\, e^T P B \right\} \Lambda \right) + 2 e^T P \xi(t) \\
&= -e^T Q e - 2 \left\| e^T P B \right\| \mathrm{trace}\left(\Delta\Theta^T \underbrace{\hat{\Theta}}_{\Theta + \Delta\Theta} \Lambda \right) + 2 e^T P \xi(t) \\
&= -e^T Q e + 2 e^T P \xi(t) \\
&\quad -2 \left\| e^T P B \right\| \mathrm{trace}\left(\Delta\Theta^T \Delta\Theta \Lambda \right) - 2 \left\| e^T P B \right\| \mathrm{trace}\left(\Delta\Theta^T \Theta \Lambda \right)
\end{aligned} \tag{11.29}$$

Using bounds (11.24) and (11.25), gives:

$$\begin{aligned}
&\dot{V} \le -\lambda_{\min}(Q) \|e\|^2 + 2\|e\| \lambda_{\max}(P) \xi_{\max} \\
&-2 \left\| e^T P B \right\| \|\Delta\Theta\|_F^2 \Lambda_{\min} + 2 \left\| e^T P B \right\| \|\Delta\Theta\|_F \|\Theta\|_F \|\Lambda\|_F \\
&= -\lambda_{\min}(Q) \|e\| \left(\|e\| - 2 \frac{\lambda_{\max}(P) \xi_{\max}}{\lambda_{\min}(Q)} \right) - 2 \left\| e^T P B \right\| \Lambda_{\min} \|\Delta\Theta\|_F \left(\|\Delta\Theta\|_F - \frac{\|\Theta\|_F \|\Lambda\|_F}{\Lambda_{\min}} \right)
\end{aligned} \tag{11.30}$$

Consequently, $\dot{V} < 0$ outside of the compact set Ω from (11.27). This argument proves UUB of all closed-loop systems trajectories, and in particular, it proves UUB tracking of the commanded signal by the system output.

For large tracking errors, the dead-zone and the e – modification slow down (dampen) the adaptation process. This undesirable effect contradicts the control goal of reducing the tracking error as fast as possible. In what follows, we will introduce the *Projection Operator*. It allows fast adaptation in the presence of parametric and nonparametric uncertainties, and at the same time the operator enforces uniform boundedness of the adaptive parameters.

12. Projection Operator

We begin with basic definitions of convex sets and functions.

Definition 12.1: A subset $\Omega \subset R^n$ is convex if

$$\left[\forall x, y \in \Omega \subset R^n \right] \Rightarrow \left[\lambda x + (1-\lambda) y = z \in \Omega \right], \quad \forall 0 \le \lambda \le 1 \tag{12.1}$$

Relation (12.1) states that if two points belong to the convex subset Ω then all the points on the connecting line also belong to Ω.

Definition 12.2: A function $f : R^n \to R$ is convex if

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall 0 \le \lambda \le 1 \tag{12.2}$$

Inequality (12.2) is illustrated in Figure 12.1. It implies that the graph of a convex function must be located below the straight line, which connects any two corresponding function values.

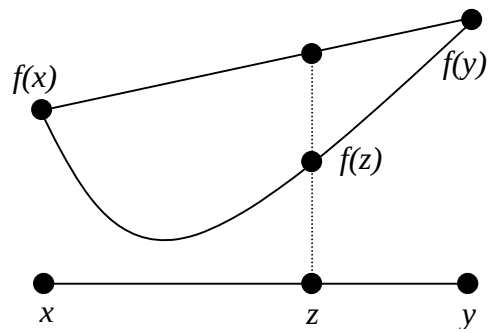

Figure 12.1: A Convex Function

Statement 12.1: Let $f(x): R^n \to R$ be convex. Then for any constant $\delta > 0$ the subset $\Omega_\delta = \{\theta \in R^n \mid f(\theta) \le \delta\}$ is convex.

Proof: Let $\theta_1, \theta_2 \in \Omega_\delta$. Then $f(\theta_1) \le \delta$ and $f(\theta_2) \le \delta$. Since $f(x)$ is convex then for any $0 \le \lambda \le 1$,

$$f\left(\underbrace{\lambda \theta_1 + (1-\lambda)\theta_2}_{\theta}\right) \le \lambda \underbrace{f(\theta_1)}_{\le\delta} + (1-\lambda) \underbrace{f(\theta_2)}_{\le\delta} \le \lambda \delta + (1-\lambda)\delta = \delta$$

Therefore, $f(\theta) \le \delta$ and, consequently, $\theta \in \Omega_\delta$ which completes the proof. □

Statement 12.2: Let $f(x): R^n \to R$ be a continuously differentiable convex function. Choose a constant $\delta > 0$ and consider the subset $\Omega_\delta = \{\theta \in R^n \mid f(\theta) \le \delta\} \subset R^n$. Let $\theta^* \in \Omega_\delta$ and assume that $f(\theta^*) < \delta$, (i.e., θ^* is <u>*not*</u> on the boundary of Ω_δ). Also, let $\theta \in \Omega_\delta$ and assume that $f(\theta) = \delta$, (i.e., (θ <u>*is*</u> on the boundary of Ω_δ). Then the following inequality takes place:

$$(\theta^* - \theta)^T \nabla f(\theta) \le 0 \tag{12.3}$$

where $\nabla f(\theta) = \begin{pmatrix} \dfrac{\partial f(\theta)}{\partial \theta_1} & \dots & \dfrac{\partial f(\theta)}{\partial \theta_n} \end{pmatrix}^T \in R^n$ is the gradient vector of f evaluated at θ.

Relation (12.3) is illustrated in Figure 12.2. It shows that the gradient vector, evaluated at the boundary of a convex set, always points away from the set.

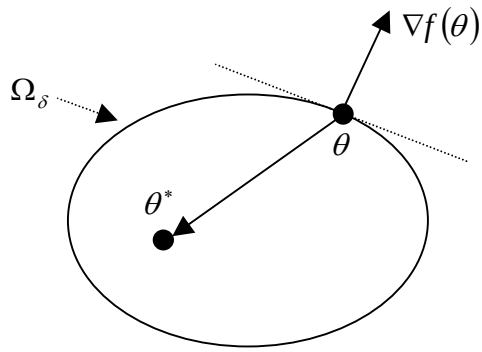

Figure 12.2: Gradient and Convex Set

Proof: Since $f(x)$ is convex then

$$f\left(\lambda\theta^* + (1-\lambda)\theta\right) \le \lambda\, f\left(\theta^*\right) + (1-\lambda)\, f(\theta)$$

or equivalently:

$$f\left(\theta + \lambda\left(\theta^* - \theta\right)\right) \le f(\theta) + \lambda\left(f\left(\theta^*\right) - f(\theta)\right)$$

Then for any nonzero $0 < \lambda \le 1$:

$$\frac{f\left(\theta + \lambda\left(\theta^* - \theta\right)\right) - f(\theta)}{\lambda} \le f\left(\theta^*\right) - f(\theta) < \delta - \delta = 0$$

Taking the limit as $\lambda \to 0$ yields relation (12.3) and completes the proof. □

Suppose that θ, the "true" parameter vector, belongs to a <u>*convex*</u> set Ω_0

$$\Omega_0 = \left\{\theta \in R^n \,\middle|\, f(\theta) \le 0\right\} \tag{12.4}$$

Introduce another convex set:

$$\Omega_1 = \left\{\theta \in R^n \,\middle|\, f(\theta) \le 1\right\} \tag{12.5}$$

It is obvious that $\Omega_0 \in \Omega_1$. We may now define the <u>*Projection Operator*</u>, which we shall use in the adaptive laws.

$$\mathrm{Proj}(\theta, y) = \begin{cases} y, & \text{if } f(\theta) \le 0 \\ y, & \text{if } f(\theta) \ge 0 \text{ and } y^T \nabla f(\theta) \le 0 \\ y - \dfrac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2}\, y\, f(\theta), & \text{if not.} \end{cases}$$

or equivalently:

$$\mathrm{Proj}(\theta, y) = \begin{cases} y - \dfrac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2}\, y\, f(\theta), & \text{if } f(\theta) > 0 \text{ and } y^T \nabla f(\theta) > 0 \\ y, & \text{if not} \end{cases} \tag{12.6}$$

By definition, $\text{Proj}(\theta, y)$ does not alter the vector y if θ belongs to the convex set Ω_0 defined in (12.4). In the set $\{0 \le f(\theta) \le 1\}$, the Projection Operator subtracts a vector normal to the boundary $\{f(\theta) = \lambda\}$ from y so that we get a smooth transformation from the original vector field y for $\lambda = 0$ to a tangent to the boundary vector field for $\lambda = 1$. The Projection Operator concept is illustrated on Figure 12.3.

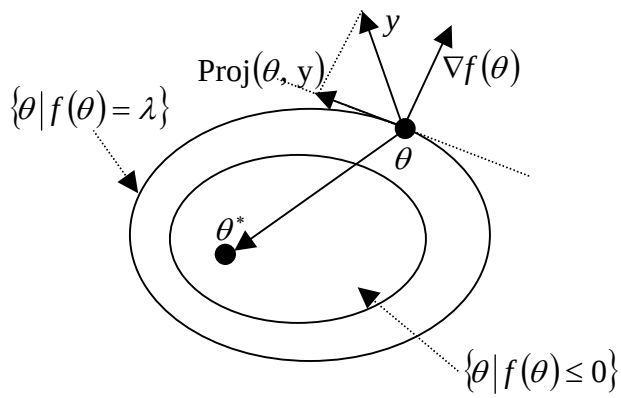

Figure 12.3: Projection Operator

Using Statement 12.2 and inequality (12.3), we get the following important property of the Projection Operator:

$$(\theta^* - \theta)^T (y - \text{Proj}(\theta, y)) = \left\{ \begin{array}{l} 0, \quad \text{if } f(\theta) \le 0 \\ 0, \quad \text{if } f(\theta) \ge 0 \text{ and } y^T \nabla f(\theta) \le 0 \\ \dfrac{\underbrace{(\theta^* - \theta)^T \nabla f(\theta)}_{\le 0} \underbrace{(\nabla f(\theta))^T y}_{\ge 0}}{\|\nabla f(\theta)\|^2} \underbrace{f(\theta)}_{=\lambda \ge 0}, \quad \text{if not.} \end{array} \right\} \le 0 \qquad (12.7)$$

or, equivalently

$$(\theta - \theta^*)^T (\text{Proj}(\theta, y) - y) \le 0 \qquad (12.8)$$

Based on (12.6), we can now define the Projection Operator when both Y and Θ are matrices of the same dimensions:

$$Y = (y_1 \quad \dots \quad y_N) \in R^{n \times N} \text{ and } \Theta = (\theta_1 \quad \dots \quad \theta_N) \in R^{n \times N} \qquad (12.9)$$

$$\text{Proj}(\Theta, Y) = (\text{Proj}(\theta_1, y_1) \quad \dots \quad \text{Proj}(\theta_N, y_N)) \qquad (12.10)$$

Thus for matrices, the Projection Operator is defined column-wise.

In the previous sections, we designed model reference adaptive control systems such that time-derivatives of the selected Lyapunov functions became negative semi-definite outside of a compact set. These time-derivatives were given n the form:

$$\dot{V} = -e^T Q e + 2\,\text{trace}\left(\Delta\Theta^T \left\{\Gamma_\Theta^{-1} \dot{\hat{\Theta}} - \Phi e^T P B\right\} \Lambda\right) + 2 e^T P \xi(t) \qquad (12.11)$$

The main task was to choose $\dot{\hat{\Theta}}$ such that the trace term in (12.11) was non-positive, and the adaptive parameters $\hat{\Theta}(t)$ became uniformly bounded functions of time.

We show how to force the trace term in (12.11) to be semi-negative by using the Projection Operator (12.10) and its property (12.8).

$$\operatorname{tr}\left(\underbrace{\Delta\Theta^T}_{\left(\hat{\Theta}-\Theta\right)^T}\left[\underbrace{\Gamma_\Theta^{-1}\dot{\hat{\Theta}}}_{\operatorname{Proj}\left(\hat{\Theta},\,Y\right)}-\underbrace{\Phi\, e^T\, P\, B}_{Y}\right]\Lambda\right)=\sum_{j=1}^{m}\underbrace{\left(\hat{\Theta}-\Theta\right)_j^T\left(\operatorname{Proj}\left(\hat{\Theta},\,Y_j\right)-Y_j\right)}_{\le 0}\underbrace{\lambda_j}_{\ge 0}\le 0 \qquad (12.12)$$

Relation (12.12) gives the following adaptive law:

$$\dot{\hat{\Theta}}=\Gamma_\Theta\operatorname{Proj}\left(\hat{\Theta},\,\Phi\, e^T\, P\, B\right) \qquad (12.13)$$

In essence, the Projection Operator ensures that the columns $\hat{\Theta}_j$ of the adaptive parameter matrix $\hat{\Theta}$ do not exceed their pre-specified bounds $\Theta_j^{\max}$. At the same time, the operator contributes to the negative semi-definiteness of the Lyapunov function (12.11).

$$\begin{aligned}\dot{V}&\le -e^T\, Q\, e+2\, e^T\, P\,\xi(t)\le -\lambda_{\min}(Q)\|e\|^2+2\|e\|\,\lambda_{\max}(P)\,\xi_{\max}\\&=-\lambda_{\min}(Q)\|e\|\left(\|e\|-2\frac{\lambda_{\max}(P)\,\xi_{\max}}{\lambda_{\min}(Q)}\right)\end{aligned} \qquad (12.14)$$

Consequently, $\dot{V}<0$ outside of the compact set,

$$\Omega=\left\{\left(e,\,\Delta\Theta\right)\in R^n\times R^{N\times m}:\quad \|e\|\le 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\xi_{\max}\wedge\|\Delta\Theta\|_F\le\Delta\Theta_{\max}\right\} \qquad (12.15)$$

where,

$$\Delta\Theta_{\max}=2\underbrace{\begin{pmatrix}\Theta_1^{\max} & \dots & \Theta_m^{\max}\end{pmatrix}}_{\Theta_{\max}}=2\,\Theta_{\max} \qquad (12.16)$$

and $\Theta_j^{\max}$ is the maximum allowable bound for the jth column $\hat{\Theta}_j(t)$. This proves UUB of all signals in the corresponding closed-loop system and, in particular, it proves UUB tracking of any external bounded command by the system regulated output.

Next we show how to define convex function $f=\begin{pmatrix}f_1 & \dots & f_m\end{pmatrix}^T$ and $m-$convex sets $\left\{\Omega_\delta^j\right\}_{j=1,\dots,m}$. Both the function and the set definitions are based on the desired upper bounds $\left\|\hat{\Theta}_j(t)\right\|\le\Theta_j^{\max}$ that are imposed column-wise. For the jth column $\hat{\Theta}_j$ of the adaptive parameter matrix $\hat{\Theta}\in R^{N\times m}$, choose <u>*projection tolerance*</u> $\varepsilon_j^\Theta>0$ and define f_j as:

$$f_j=f\left(\hat{\Theta}_j\right)=\frac{\left\|\hat{\Theta}_j\right\|^2-\left(\Theta_j^{\max}\right)^2}{\varepsilon_j^\Theta\left(\Theta_j^{\max}\right)^2} \qquad (12.17)$$

Using (12.17), the sets Ω_δ^j are defined as:

$$\Omega_\delta^j = \left\{ \hat{\Theta}_j \in R^{N\times 1} \,\middle|\, f_j \le \delta_j \right\} \tag{12.18}$$

From (12.18) it follows that for each $j = 1, \ldots, m$:

$$\begin{aligned} \Omega_0^j &= \left\{ \hat{\Theta}_j \in R^{N\times 1} : \quad \left\| \hat{\Theta}_j \right\| \le \Theta_j^{\max} \right\} \\ \Omega_1^j &= \left\{ \hat{\Theta}_j \in R^{N\times 1} : \quad \left\| \hat{\Theta}_j \right\| \le \Theta_j^{\max} \sqrt{1+\varepsilon_j^x} \right\} \end{aligned} \tag{12.19}$$

The gradient of the convex function (12.17) can be computed as:

$$\nabla f_j = \frac{1}{\varepsilon_j^\Theta \left(\Theta_j^{\max}\right)^2} \nabla \left[\left\| \hat{\Theta}_j \right\|^2 \right] = \frac{2}{\varepsilon_j^\Theta \, \Theta_j^{\max}} \hat{\Theta}_j \tag{12.20}$$

Using (12.13), the adaptive law for $\hat{\Theta}_j$ becomes:

$$\dot{\hat{\Theta}}_j = \Gamma_\Theta \begin{cases} \left(\Phi e^T P B\right)_j + \dfrac{\nabla f_j \nabla f_j^T}{\left\| \nabla f_j \right\|^2} \left(\Phi e^T P B\right)_j f_j, & \text{if } \begin{cases} f_j > 0 \\ \left(\Phi e^T P B\right)_j^T \nabla f_j < 0 \end{cases} \\ \left(\Phi e^T P B\right)_j & , \quad \text{if not} \end{cases} \tag{12.21}$$

The adaptation process in (12.21) ensures uniform boundedness of each column of the adaptive time-dependent parameter matrix $\hat{\Theta}(t)$ forward in time, that is:

$$\left\{ \left\| \hat{\Theta}_j(0) \right\| \le \Theta_j^{\max} \right\} \Rightarrow \left\{ \left\| \hat{\Theta}_j(t) \right\| \le \Theta_j^{\max} \sqrt{\left(1+\varepsilon_j^\Theta\right)}, \quad \forall t \ge 0, \quad 1 \le j \le m \right\} \tag{12.22}$$