Lecture 5

9. MRAC with Integral Action

In this section, we design tracking systems using integral control. We begin by considering a class of MIMO uncertain systems in the form:

$$\dot{x}_p = A_p x_p + B_p \Lambda \left( u + \Theta_d^T \Phi_d (x_p) \right)$$  \hspace{1cm} (9.1)

where $x_p \in X_p \subset \mathbb{R}^{n_p}$ is the system state vector, $u \in U \subset \mathbb{R}^m$ is the control input,

$$d(x_p) = \Theta_d^T \Phi_d (x_p)$$  \hspace{1cm} (9.2)

is the linear-in-parameters state-dependent matched uncertainty, $\Theta_d \in \mathbb{R}^{N \times m}$ is the matrix of unknown constant parameters, and $\Phi_d (x_p) \in \mathbb{R}^N$ is the known $N \times 1$ dimensional regressor vector, whose components are locally Lipschitz continuous functions of $x_p$. Also in (9.1), $B_p \in \mathbb{R}^{n \times m}$ is constant and known, $A_p \in \mathbb{R}^{n \times n}$ is constant and unknown, and $\Lambda \in \mathbb{R}^{m \times m}$ is constant, diagonal and unknown matrix with positive diagonal elements.

The control goal of interest is bounded tracking, that is one needs to define $u$ such that the system controlled output

$$y = C_p x_p \in \mathbb{R}^m$$  \hspace{1cm} (9.3)

tracks any bounded and possibly time-varying command $r(t) \in \mathbb{R}^m$, in the presence of the system uncertainties $\{A_p, \Lambda, \Theta_d\}$. It is assumed that $C_p$ is known. Let,

$$e_y(t) = y(t) - r(t)$$  \hspace{1cm} (9.4)

denote the system output tracking error. Augmenting (9.1) with the integrated output tracking error,

$$e_{y, s}(t) = \int_0^t e_y(\tau) d\tau \leftrightarrow \begin{pmatrix} e_{y, s} \\ e_y \end{pmatrix}$$  \hspace{1cm} (9.5)

yields the extended open-loop system dynamics:

$$\dot{x} = Ax + B \Lambda \left( u + d(x_p) \right) + B_c r$$  \hspace{1cm} (9.6)

where $x = \begin{pmatrix} e_{y, s}^T & x_p^T \end{pmatrix} \in \mathbb{X} \subset \mathbb{R}^n$ is the extended system state vector, whose dimension is $n = n_p + m$. The system matrices $(A, B, B_c)$ are

$$A = \begin{pmatrix} 0_{m \times m} & C_p \\ 0_{n_p \times m} & A_p \end{pmatrix}, \hspace{1cm} B = \begin{pmatrix} 0_{m \times m} \\ B_p \end{pmatrix}, \hspace{1cm} B_c = \begin{pmatrix} -I_{m \times m} \\ 0_{n_p \times m} \end{pmatrix}$$  \hspace{1cm} (9.7)

and

$$y = \begin{pmatrix} 0_{m \times m} & C_p \end{pmatrix} \dot{x} = C \dot{x}$$  \hspace{1cm} (9.8)
represents the system controlled output.

To summarize, in this section we consider MIMO uncertain dynamical systems in the form of (9.6):

$$\dot{x} = Ax + B \Lambda (u + \Theta^T_d \Phi_d (x_p)) + B_c r(t)$$  \hspace{1cm} (9.9)

$$y = C x$$

with an unknown constant matrix $A \in \mathbb{R}^{n \times n}$, known matrices $B \in \mathbb{R}^{n \times m}$, $B_c \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, an unknown constant diagonal positive-definite matrix $\Lambda \in \mathbb{R}^{m \times m}$, and an unknown matrix of constant parameters $\Theta_d \in \mathbb{R}^{N \times m}$. The state-dependent regressor vector $\Phi_d (x_p) \in \mathbb{R}^N$ is locally Lipschitz-continuous in $x_p$ and known. The control goal is bounded tracking of any bounded time-varying reference signal $r(t) \in \mathbb{R}^{m_1}$ by the system output $y(t) \in \mathbb{R}^{m_1}$, in the presence of constant parametric uncertainties $(A, \Lambda, \Theta_d)$.

**Assumption 9.1 (Model Matching Conditions)**

Given a reference Hurwitz matrix $A_{ref}$ and an unknown positive-definite diagonal constant matrix $\Lambda$, there must exist a constant, possibly unknown, gain matrix $K_x$ such that

$$A_{ref} = A + B \Lambda K_x^T$$  \hspace{1cm} (9.10)

Using the matching condition (9.10), we rewrite the system dynamics in the form:

$$\dot{x} = A_{ref} x + B \Lambda (u - K_x^T x + \Theta_d^T \Phi_d (x_p)) + B_c r$$  \hspace{1cm} (9.11)

Control input $u$ is chosen as:

$$u = \hat{K}_x^T x - \hat{\Theta}_d^T \Phi_d (x_p)$$  \hspace{1cm} (9.12)

where $\hat{K}_x(t) \in \mathbb{R}^{n \times m}$ and $\hat{\Theta}_d(t) \in \mathbb{R}^{N \times m}$ are adaptive time-varying matrices, whose dynamics will be defined later. Substituting (9.12) into (9.11), yields:

$$\dot{x} = A_{ref} x + B \Lambda \left( \frac{\hat{K}_x - K_x}{\Delta K_x} x - \frac{\hat{\Theta}_d - \Theta_d}{\Delta \Theta_d} \Phi_d (x_p) \right) + B_c r$$  \hspace{1cm} (9.13)

Based on (9.13), consider the following reference model:

$$\dot{x}_{ref} = A_{ref} x_{ref} + B_c r$$  \hspace{1cm} (9.14)

and let

$$e = x - x_{ref}$$  \hspace{1cm} (9.15)

represent the system tracking error. Subtracting (9.14) from (9.13) results in the tracking error dynamics,
\[ \dot{e} = A_{\text{ref}} e + B \Lambda \left( \Delta K^T x - \Delta \Theta_d^T \Phi_d \left( x_p \right) \right) \]  

(9.16)

Consider the following Lyapunov function candidate,

\[ V(e, \Delta K_x, \Delta \Theta_d) = e^T P_{\text{ref}} e + \text{trace} \left( \Delta K^T \Gamma^{-1} \Delta K_x \Lambda \right) + \text{trace} \left( \Delta \Theta_d^T \Gamma_{\Theta_d}^{-1} \Delta \Theta_d \Lambda \right) \]  

(9.17)

where \( \Gamma_x = \Gamma_x^T > 0 \) and \( \Gamma_{\Theta_d} = \Gamma_{\Theta_d}^T > 0 \) denote constant rates of adaptation, and \( P_{\text{ref}} = P_{\text{ref}}^T > 0 \) is the unique symmetric positive definite solution of the algebraic Lyapunov equation,

\[ A_{\text{ref}}^T P_{\text{ref}} + P_{\text{ref}} A_{\text{ref}} = -Q_{\text{ref}} \]  

(9.18)

with \( Q_{\text{ref}} = Q_{\text{ref}}^T > 0 \). Time-differentiating \( V \), along the trajectories of (9.16), gives:

\[ \dot{V} = -e^T Q_{\text{ref}} e + 2 e^T P_{\text{ref}} B \Lambda \left( \Delta K^T \Gamma^{-1} \dot{K}_x \Lambda \right) + 2 e^T P_{\text{ref}} B \Lambda \]  

(9.19)

Applying the trace identity (valid for any two co-dimensional vectors \( a \) and \( b \)),

\[ a^T b = \text{trace} \left( b a^T \right) \]  

(9.20)

yields:

\[ \dot{V} = -e^T Q_{\text{ref}} e + 2 \text{trace} \left( \Delta K^T \Gamma^{-1} \dot{K}_x \Lambda \right) \]  

(9.21)

If adaptive laws are selected in the form,

\[ \dot{K}_x = -\Gamma_x x e^T P_{\text{ref}} B \]  

\[ \dot{\Theta}_d = \Gamma_{\Theta_d} \Phi_d e^T P_{\text{ref}} B \]  

(9.22)

then,

\[ V(e, \Delta K_x, \Delta \Theta_d) = -e^T Q_{\text{ref}} e \leq 0 \]  

(9.23)

which in turn, proves UUB of \( (e, \Delta K_x, \Delta \Theta_d) \). Moreover, one can show that the tracking error signal is square integrable: \( e \in L^2 \). Since \( r \in L^\infty \) then \( x_{\text{ref}} \in L^\infty \), and consequently, \( x \in L^\infty \). Since the ideal (unknown) parameters \( (K_x, \Theta_d) \) are constant and their estimation errors \( (\Delta K_x, \Delta \Theta_d) \) are bounded, then their estimated values are bounded as well, that is: \( \hat{K}_x, \hat{\Theta}_d \in L^\infty \). Since components of the regressor vector \( \Phi_d \left( x_p \right) \) are locally Lipschitz -continuous functions of \( x_p \), and the latter is bounded, then the components themselves are bounded. Hence, \( u \in L^\infty \) and \( \dot{\chi} \in L^\infty \). Thus \( \dot{e} \in L^\infty \), which implies that \( \dot{v} \in L^\infty \). Therefore, \( \dot{V} \) is a uniformly continuous function of time. Since \( V \) is lower bounded, \( \dot{V} \leq 0 \), and \( \dot{V} \) is uniformly continuous then \( V \) tends to a limit, while its derivative \( \dot{V} \) tends to zero, (due to Barbalat’ Lemma). Consequently, the tracking error, \( e \) tends to zero asymptotically, as \( t \to \infty \). Moreover, since the Lyapunov function (9.17) is radially unbounded, then the asymptotic convergence is global, that is the closed-loop tracking error dynamics are globally asymptotically stable.
We have shown that the system state $x$ globally asymptotically tracks the state $x_{ref}$ of the reference model, and therefore the system output $y = Cx$ globally asymptotically tracks the reference model output $y_{ref} = Cx_{ref}$. At the same time, the reference model dynamics are chosen such that $y_{ref}$ tracks an external bounded command $r(t)$, with bounded errors. Therefore, $y$ must also track $r$ with bounded errors. The tracking problem is solved. The end result is summarized below.

**Theorem 9.1**

Consider the uncertain system dynamics in (9.9), operating under the MRAC controller (9.12), with the adaptive laws (9.22). Suppose that the matching relations (9.10) hold. Let the reference model (9.14) be driven by a bounded external command $r(t)$. Then for any symmetric positive definite matrices $(\Gamma, \Gamma_{\phi}, Q_{ref})$, all signals in the closed-loop system,

$$
\begin{align*}
\dot{x} &= Ax + B \Lambda \left( \hat{K}_x^T x - \hat{\Theta}_d^T \Phi_d \left( x_p \right) + \Theta_d^T \Phi_d \left( x_p \right) \right) + B_c r(t) \\
\dot{x}_{ref} &= A_{ref} x_{ref} + B_c r(t) \\
\dot{\hat{K}}_x &= -\Gamma_x \left( x - x_{ref} \right)^T P_{ref} B \\
\dot{\hat{\Theta}}_d &= \Gamma_{\phi_d} \Phi_d \left( x_p \right) \left( x - x_{ref} \right)^T P_{ref} B
\end{align*}
$$

(9.24)

are uniformly ultimately bounded in time, where symmetric positive definite matrix $P_{ref}$ is the unique solution of the algebraic Lyapunov equation (9.18). Moreover, the tracking error signal $e = x - x_{ref}$ is UUB, square integrable, and globally asymptotically stable,

$$
\lim_{t \to \infty} \|e(t)\| = 0.
$$

**Remark 9.1**

As it was shown in the previous section, the MRAC design methodology can be applied to the original system (9.1), and without adding integrated output tracking errors. In this case, the control input is:

$$
u = \hat{K}_r^T x + \hat{\Theta}_d^T r - \hat{\Theta}_d^T \Phi_d \left( x_p \right)
$$

(9.25)

where $\hat{K}_r \in R^{m \times m}$ is the estimated feedforward gain. The system closed-loop dynamics become:
\[
\dot{x} = A_{ref} x + B \Lambda \begin{pmatrix}
\frac{\Delta K_x^T}{\Delta K_x} x \\
\frac{\Delta K_r^T}{\Delta K_r} x \\
\frac{\Delta \Theta_d^T}{\Delta \Theta_d} \Theta_d(x_p)
\end{pmatrix} + B \Lambda K_r^T r
\]

\[= A_{ref} x + B \Lambda \left( \Delta K_x^T x - \Delta \Theta_d^T \Phi_d(x_p) \right) + B_r r\]

where \(K_r \in R^{m \times m}\) is the unknown / ideal feedforward gain. The relation \(B_r = B \Lambda K_r^T\) is the additional required matching condition, which guarantees existence of the ideal feedforward gain \(K_r\). The corresponding adaptive laws are:

\[
\dot{K}_x = -\Gamma_x x e^T P_{ref} B
\]

\[
\dot{K}_r = -\Gamma_r r e^T P_{ref} B
\]

\[
\dot{\Theta}_d = \Gamma_\Theta \Phi_d e^T P_{ref} B
\]

where \(\Gamma_r = \Gamma_r^T > 0\) is the adaptation rate matrix for the feedforward adaptive gain \(\hat{K}_r\).

**10. MRAC Augmentation of an Optimal Baseline Controller with Integral Action**

The adaptive design developed in the previous section can be applied to *augment* a baseline linear controller with (Proportional + Integral) (PI) feedback architecture. The rational for using an augmentation approach (as oppose to all adaptive) stems from the fact that in most realistic applications, a system may already have a baseline controller. This controller would have been designed to operate under nominal conditions (no uncertainties). If adding uncertainties destroys the desired baseline closed-loop performance then one might attempt to recover the desired nominal performance by augmenting the baseline controller with an adaptive element.

We consider a class of multi-input-multi-output (MIMO) nonlinear systems, whose plant dynamics are linearly parameterized, the uncertainties satisfy the matching conditions, and the system state is measurable, (i.e., available on-line as the system output). More specifically, we consider \(n^{th}\) order MIMO systems in the form,

\[
\dot{x}_p = A_p x_p + B_p \Lambda \left( u + f(x_p) \right)
\]

where \(n_p\) and \(m\) denote dimensions of the system state \(x_p\) and control \(u\), respectively. Also, \(x_p \in R^{n_p}\) is the system state, \(u \in R^m\) is the control input, \(A_p \in R^{n_p \times n_p}\) and \(B_p \in R^{m \times m}\) are known, while \(\Lambda \in R^{m \times m}\) is unknown. In addition, it is assumed that \(\Lambda\) is diagonal, its elements \(\lambda_i\) are positive, and the pair \((A_p, (B_p \Lambda))\) is controllable. The uncertainty \(\Lambda\) is introduced to model control failures.
Moreover, the unknown nonlinear function $f(x_p): \mathbb{R}^n \rightarrow \mathbb{R}^m$ represents the system matched uncertainty. It is assumed that this function can be written as a linear combination of $N$ known basis functions, with unknown constant coefficients.

$$f(x_p) = \Theta^T \Phi(x_p)$$  \hspace{1cm} (10.2)

In (10.2), $\Theta \in \mathbb{R}^{N \times m}$ is the unknown constant matrix of ideal parameters, while $\Phi(x_p) \in \mathbb{R}^N$ represents the known locally Lipschitz-continuous regressor vector. Thus, we consider MIMO systems of the form:

$$\dot{x}_p = A_p x_p + B_p \Lambda \left( u + \Theta^T \Phi(x_p) \right)$$  \hspace{1cm} (10.3)

with the controlled output

$$y = C_p x_p + D_p \Lambda \left( u + \Theta^T \Phi(x_p) \right)$$  \hspace{1cm} (10.4)

Let $y_{cmd}(t) \in \mathbb{R}^m$ denote a bounded command for the system controlled output $y \in \mathbb{R}^m$ to follow. This task is to be accomplished using the system control input $u \in \mathbb{R}^m$, in the form of a full state feedback. We define output tracking error $e_y$, its integral $e_{yI}$,

$$\dot{e}_{yI} = e_y = y - y_{cmd}$$  \hspace{1cm} (10.5)

and consider the extended open-loop dynamics:

$$\begin{pmatrix} \dot{e}_{yI} \\ \dot{x}_p \end{pmatrix} = \begin{pmatrix} 0_{n_{x,m}} & C_p \\ 0_{n_{x,m}} & A_p \end{pmatrix} \begin{pmatrix} e_{yI} \\ x_p \end{pmatrix} + \begin{pmatrix} D_p \\ B_p \end{pmatrix} \Lambda \left( u + f(x_p) \right) + \begin{pmatrix} -I_{m_{x,m}} \\ 0_{n_{x,m}} \end{pmatrix} y_{cmd}$$  \hspace{1cm} (10.6)

or, equivalently

$$\dot{x} = Ax + B \Lambda \left( u + \Theta^T \Phi(x_p) \right) + B_{ref} y_{cmd}$$  \hspace{1cm} (10.7)

In terms of (10.7), the controlled output $y$ from (10.4) can be written as:

$$y = \begin{pmatrix} 0 & C_p \end{pmatrix} \begin{pmatrix} e_{yI} \\ x_p \end{pmatrix} + D_p \Lambda \left( u + \Theta^T \Phi(x_p) \right) u = C x + D \Lambda \left( u + \Theta^T \Phi(x_p) \right) u$$  \hspace{1cm} (10.8)

The control problem of interest is bounded tracking in the presence of the system constant parametric uncertainties $\Lambda$ and $\Theta$. Specifically, one needs to design the control input $u$, so that the system controlled output $y$ tracks any bounded time-varying command $y_{cmd}$, with bounded tracking errors, while the rest of the signals in the corresponding closed-loop dynamics remain bounded.

We begin with the design of a baseline linear controller. Setting $\Lambda = I_{m_{x,m}}$, $\Theta = 0_{N \times m}$ in (10.7), results in the linear time invariant (LTI) baseline dynamics.

$$\dot{x} = Ax + Bu + B_{ref} y_{cmd}$$
$$y = Cx + Du$$  \hspace{1cm} (10.9)
Assuming constant command \( y_{cmd} \), we can use the Linear Quadratic Regulator (LQR) method, with Proportional + Integral (PI) feedback connections, to design a baseline optimal control law. This controller represents the so-called LQR PI servomechanism. Its design is outlined below.

We first calculate an optimal stabilizing controller for:
\[
\dot{z} = Az + Bv
\]  
where
\[
z = \dot{x} = \begin{pmatrix} \dot{y}_I \\ \dot{x}_p \end{pmatrix}, \quad v = \dot{u}
\]  
and the control input is designed to minimize the Linear Quadratic cost:
\[
J(v) = \int_0^\infty \left( z^T Q z + v^T R v \right) dt \rightarrow \min
\]  
with the appropriately selected symmetric positive definite matrices \( Q \) and \( R \). It is well-known that the corresponding optimal LQR solution is given in feedback form:
\[
\dot{u} = v = -R^{-1}B^T P z = -\begin{pmatrix} K_I & K_P \end{pmatrix} \begin{pmatrix} \dot{y}_I \\ \dot{x}_p \end{pmatrix}
\]  
In (10.13), \( P \) is the unique symmetric positive semi-definite solution of the algebraic Riccati equation,
\[
A^T P + PA + Q - PBR^{-1}B^T P = 0
\]  
which is solved using appropriately chosen \( Q = Q^T \geq 0 \). Integrating (10.13), yields the baseline LQR PI controller,
\[
u_{bl} = -K_x^T x = -K_I e_I - K_P x = K_I \left( \frac{y_{cmd} - y}{s} \right) - K_P x_p
\]  
where the optimal gain matrix
\[
K_x^T = \begin{pmatrix} K_I & K_P \end{pmatrix}
\]  
is partitioned into the integral gain \( K_I \) and the proportional gain \( K_P \). The corresponding control block-diagram is shown in Figure 11.1.

In the presence of the system uncertainties \( \Lambda \) and \( \Theta \), the baseline tracking performance will often deteriorate. In order to restore the desired baseline behavior, an adaptive augmentation can be added. This process consists of: a) A reference model definition, b) Tracking dynamics formulation, and b) The design of adaptive laws.

Figure 11.1. Baseline / Servomechanism LQR PI Control Block-Diagram
We define the reference model to represent the baseline closed-loop system dynamics, which are obtained by substituting the baseline controller (10.15) into the LTI system (10.9). The resulting reference model dynamics are of the form:

\[
\begin{align*}
\dot{x}_{\text{ref}} &= A_{\text{ref}} x_{\text{ref}} + B_{\text{ref}} y_{\text{cmd}} \\
y_{\text{ref}} &= C_{\text{ref}} x_{\text{ref}}
\end{align*}
\]  
(10.17)

where

\[
\begin{align*}
A_{\text{ref}} &= A - BK^*_x \\
C_{\text{ref}} &= C - DK^*_x
\end{align*}
\]  
(10.18)

and \(A_{\text{ref}}\) is Hurwitz by design. It is easy to see that the DC gain from the command \(y_{\text{cmd}}\) to the reference output \(y_{\text{ref}}\) is unity, (Homework: Prove this statement).

Total control input is defined as the sum of the baseline LQR PI component (10.15) and its adaptive augmentation:

\[
u = -K^*_x x + u_{ad} = u_{bl} + u_{ad}
\]  
(10.19)

Substituting (10.19) into the original system dynamics (10.7), gives:

\[
\begin{align*}
\dot{x} &= A_{\text{ref}} x + B A \left( u_{ad} + \left( I_{mxm} - \Lambda^{-1} \right) u_{bl} + \Theta^T \Phi(x_p) \right) + B_{\text{ref}} y_{\text{cmd}} \\
y &= C_{\text{ref}} x + D A \left( u_{ad} + \Theta^T \Phi(u_{bl}, x_p) \right)
\end{align*}
\]  
(10.20)

or equivalently,

\[
\begin{align*}
\dot{x} &= A_{\text{ref}} x + B A \left( u_{ad} + \Theta^T \Phi(u_{bl}, x_p) \right) + B_{\text{ref}} y_{\text{cmd}} \\
y &= C_{\text{ref}} x + D A \left( u_{ad} + \Theta^T \Phi(u_{bl}, x_p) \right)
\end{align*}
\]  
(10.21)

with the redefined regressor vector

\[
\Phi(u_{bl}, x_p) = \left( u_{bl}^T \Phi(x_p) \right)^T
\]  
(10.22)

and with the extended matrix of unknown / ideal parameters.

\[
\Theta = \left( K^*_u \quad \Theta^T \right)^T
\]  
(10.23)

The adaptive component \(u_{ad}\) is chosen to dominate / approximate the system matched uncertainty \(\Theta^T \Phi(x)\).

\[
u_{ad} = -\hat{\Theta}^T \Phi(u_{bl}, x_p)
\]  
(10.24)

where \(\hat{\Theta} \in R^{(n+N) \times m}\) is the matrix of adaptive parameters. Substituting (10.24) into (10.21), results in:
\[
\dot{x} = A_{\text{ref}} x - B \Lambda \left( \hat{\Theta} - \bar{\Theta} \right)^T \Phi + B_{\text{cmd}} y_{\text{cmd}} \\
y = C_{\text{ref}} x - D \Delta \bar{\Theta}^T \Phi
\]

(10.25)

where,
\[
\Delta \bar{\Theta} = \hat{\Theta} - \bar{\Theta}
\]

(10.26)
is the matrix of *parameter estimation errors*. Introduce the *tracking error*:
\[
e = x - x_{\text{ref}}
\]

(10.27)

Then the *tracking error dynamics* can be calculated by subtracting the reference system dynamics (10.17) from the open-loop system dynamics (10.25).
\[
\dot{e} = A_{\text{ref}} e - B \Lambda \Delta \bar{\Theta}^T \Phi
\]

(10.28)

Consider the Lyapunov function candidate:
\[
V(e, \Delta \bar{\Theta}) = e^T P_{\text{ref}} e + \text{trace} \left( \Delta \bar{\Theta}^T \Gamma_{\Theta}^{-1} \Delta \bar{\Theta} \Lambda \right)
\]

(10.29)

where \(\Gamma_{\Theta} = \Gamma_{\Theta}^T > 0\) denotes constant rates of adaptation, and \(P_{\text{ref}} = P_{\text{ref}}^T > 0\) is the unique symmetric positive definite solution of the algebraic Lyapunov equation,
\[
A_{\text{ref}}^T P_{\text{ref}} + P_{\text{ref}} A_{\text{ref}} = -Q_{\text{ref}}
\]

(10.30)

with \(Q_{\text{ref}} = Q_{\text{ref}}^T > 0\). Time-differentiating \(V\), along the trajectories of (10.28), gives:
\[
\dot{V} = -e^T Q_{\text{ref}} e - 2 e^T P_{\text{ref}} B \Lambda \Delta \bar{\Theta}^T \Phi + 2 \text{trace} \left( \Delta \bar{\Theta}^T \Gamma_{\Theta}^{-1} \hat{\Theta} \Lambda \right)
\]

(10.31)

Applying the trace identity,
\[
a^T b = \text{trace}(b a^T)
\]

(10.32)
yields:
\[
\dot{V} = -e^T Q_{\text{ref}} e + 2 \text{trace} \left( \Delta \bar{\Theta}^T \left( \Gamma_{\Theta}^{-1} \hat{\Theta} - \bar{\Theta} e^T P_{\text{ref}} B \right) \Lambda \right)
\]

(10.33)

If adaptive laws are selected in the form,
\[
\hat{\Theta} = \Gamma_{\Theta} \bar{\Theta} \left( u_{bl}, x_p \right) e^T P_{\text{ref}} B
\]

(10.34)

then,
\[
V(e, \Delta \bar{\Theta}) = -e^T Q_{\text{ref}} e \leq 0
\]

(10.35)

which in turn, proves UUB of \((e, \Delta \bar{\Theta})\).

Moreover, it follows from (10.35) that the tracking error signal is square integrable, \(e \in L_2\). Since \(y_{\text{cmd}} \in L_\infty\) then \(x_{\text{ref}} \in L_\infty\), and consequently, \(x \in L_\infty\) and \(\left( u_{bl}, x_p \right) \in L_\infty\). Since the ideal (unknown) matrix of parameters \(\bar{\Theta}\) is constant and the estimation errors \(\Delta \bar{\Theta}\) are bounded, then their estimated values are bounded as well, that is: \(\hat{\Theta} \in L_\infty\). Since components of the regressor vector \(\Phi \left( u_{bl}, x_p \right)\) are locally Lipschitz -continuous, and \(\left( u_{bl}, x_p \right) \in L_\infty\) then the regressor components are bounded. Hence, \(u \in L_\infty\) and \(\dot{x} \in L_\infty\).
Thus $\dot{e} \in L_w$, which implies that $\dot{V} \in L_\infty$. Therefore, $\dot{V}$ is a uniformly continuous function of time. Since $V$ is lower bounded, $\dot{V} \leq 0$, and $\dot{V}$ is uniformly continuous then $V$ tends to a limit, while its derivative $\dot{V}$ tends to zero, (due to Barbalat’ Lemma). Consequently, the tracking error $e$ tends to zero asymptotically, as $t \to \infty$. Moreover, since the Lyapunov function (10.29) is radially unbounded, then the asymptotic convergence is global, that is the closed-loop tracking error dynamics (10.28) are globally asymptotically stable. Consequently, $\lim_{t \to \infty} \| \Delta \dot{\Phi}^T \Phi(t) \dot{\Phi}(u_{bl}(t), x(t)) \| = 0$ and,

$$y = C x - D \Delta \Theta^T \Phi \to C_{ref} x \to C_{ref} x_{ref} = y_{ref}$$

(10.36)

that is for any bounded command $y_{cmd}$, the closed-loop system output from (10.25) globally asymptotically tracks the reference model output from (10.17), as $t \to \infty$. At the same time, the reference model dynamics (10.17) are chosen such that $y_{ref}$ tracks an external bounded command $y_{cmd}(t)$, with bounded errors. Therefore, $y$ must also track $y_{cmd}$ with bounded errors. The tracking problem is solved.

**Remark 10.1**

The adaptive laws (10.34) can be written in terms of the system original parameters. Let’s partition the adaptive rate matrix,

$$\Gamma_{\tilde{\Theta}} = \begin{pmatrix} \Gamma_u & 0_{nxm} \\ 0_{Nx,m} & \Gamma_{\Theta} \end{pmatrix}$$

(10.37)

where $(\Gamma_u, \Gamma_{\Theta})$ denote the adaptive rates for uncertainties that correspond to $x$ and $\Phi(x_p)$. Using (10.22), (10.23), (10.37), the adaptive laws (10.34) can now be written as:

$$\dot{K}_u = \Gamma_u u_{bl} e^T P_{ref} B$$

$$\dot{\Theta} = \Gamma_{\Theta} \Phi(x_p) e^T P_{ref} B$$

(10.38)

Also, the (LQR PI Baseline + Adaptive) total control input (10.19) becomes,

$$u = u_{bl} + u_{ad} = \left[ -K_x^T x \right]_{u_{bl} \text{ - Baseline}} + \left[ -K_u u_{bl} ^T - \tilde{\Theta}^T \Phi(x_p) \right]_{u_{ad} \text{ - Adaptive Augmentation}}$$

(10.39)

or, equivalently

$$u = \left( I_{nxm} - \hat{K}_u^T \right) u_{bl} - \tilde{\Theta}^T \Phi(x_p) = - \left( I_{nxm} - \hat{K}_u^T \right) K_x^T x - \tilde{\Theta}^T \Phi(x_p)$$

$$= \left( I_{nxm} - \hat{K}_u^T \right) \left( K_x \frac{y_{cmd} - y}{s} - K_p x_p \right) - \tilde{\Theta}^T \Phi(x_p)$$

(10.40)

Note that by the design, this controller does not have a feedforward component. Also note that in the adaptive laws (10.38), the adaptive parameter initial values can be set to zero.