Lecture 5

9. MRAC with Integral Action

In this section, we design tracking systems using *integral* control. We begin by considering a class of MIMO *uncertain* systems in the form:

$$\dot{x}_p = A_p\, x_p + B_p\, \Lambda \left(u + \overbrace{\Theta_d^T\, \Phi_d\left(x_p\right)}^{d\left(x_p\right)} \right) \tag{9.1}$$

where $x_p \in X_p \subset R^{n_p}$ is the system state vector, $u \in U \subset R^m$ is the control input,

$$d\left(x_p\right) = \Theta_d^T\, \Phi_d\left(x_p\right) \tag{9.2}$$

is the linear-in-parameters state-dependent matched uncertainty, $\Theta_d \in R^{N\times m}$ is the matrix of unknown constant parameters, and $\Phi_d\left(x_p\right) \in R^N$ is the known N – dimensional regressor vector, whose components are locally Lipschitz -continuous functions of x_p. Also in (9.1), $B_p \in R^{n\times m}$ is constant and *known*, $A_p \in R^{n\times n}$ is constant and *unknown*, and $\Lambda \in R^{m\times m}$ is constant, diagonal and *unknown* matrix with positive diagonal elements.

The *control goal* of interest is *bounded tracking*, that is one needs to define u such that the system controlled output

$$y = C_p\, x_p \in R^m \tag{9.3}$$

tracks any bounded and possibly time-varying command $r\left(t\right) \in R^m$, in the presence of the system uncertainties $\left\{A_p,\, \Lambda,\, \Theta_d\right\}$. It is assumed that C_p is known. Let,

$$e_y\left(t\right) = y\left(t\right) - r\left(t\right) \tag{9.4}$$

denote the system *output tracking error*. Augmenting (9.1) with the *integrated output tracking error*,

$$\left(e_{yI}\left(t\right) = \int_0^t e_y\left(\tau\right) d\tau \right) \Leftrightarrow \left(e_{yI} = \frac{e_y}{s} \right) \tag{9.5}$$

yields the *extended* open-loop system dynamics:

$$\dot{x} = A\,x + B\,\Lambda\left(u + d\left(x_p\right)\right) + B_c\, r \tag{9.6}$$

where $x = \left(e_{yI}^T \quad x_p^T\right) \in X \subset R^n$ is the extended system state vector, whose dimension is $n = n_p + m$. The system matrices $\left(A,\, B,\, B_c\right)$ are

$$A = \begin{pmatrix} 0_{m\times m} & C_p \\ 0_{n_p\times m} & A_p \end{pmatrix},\; B = \begin{pmatrix} 0_{m\times m} \\ B_p \end{pmatrix},\; B_c = \begin{pmatrix} -I_{m\times m} \\ 0_{n_p\times m} \end{pmatrix} \tag{9.7}$$

and

$$y = \underbrace{\left(0_{m\times m} \quad C_p\right)}_{C} x = C\,x \tag{9.8}$$

represents the system *controlled output*.

To summarize, in this section we consider MIMO *uncertain* dynamical systems in the form of (9.6):

$$\begin{aligned} \dot{x} &= A\,x + B\,\Lambda\left(u + \Theta_d^T\,\Phi_d\left(x_p\right)\right) + B_c\,r\left(t\right) \\ y &= C\,x \end{aligned} \tag{9.9}$$

with an unknown constant matrix $A \in R^{n\times n}$, known matrices $B \in R^{n\times m}$, $B_c \in R^{n\times m}$, $C \in R^{m\times n}$, an unknown constant diagonal positive-definite matrix $\Lambda \in R^{m\times m}$, and an unknown matrix of constant parameters $\Theta_d \in R^{N\times m}$. The state-dependent regressor vector $\Phi_d\left(x_p\right) \in R^N$ is locally Lipschitz -continuous in x_p and known. The *control goal* is *bounded tracking* of any bounded time-varying reference signal $r\left(t\right) \in R^{m\times 1}$ by the system output $y\left(t\right) \in R^{m\times 1}$, in the presence of constant parametric uncertainties $\left(A,\ \Lambda,\ \Theta_d\right)$.

Assumption 9.1 (Model Matching Conditions)

Given a reference Hurwitz matrix A_{ref} and an unknown positive-definite diagonal constant matrix Λ, there must exist a constant, possibly unknown, gain matrix K_x such that

$$A_{ref} = A + B\,\Lambda\,K_x^T \tag{9.10}$$

Using the matching condition (9.10), we rewrite the system dynamics in the form:

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(u - K_x^T\,x + \Theta_d^T\,\Phi_d\left(x_p\right)\right) + B_c\,r \tag{9.11}$$

Control input u is chosen as:

$$u = \hat{K}_x^T\,x - \hat{\Theta}_d^T\,\Phi_d\left(x_p\right) \tag{9.12}$$

where $\hat{K}_x\left(t\right) \in R^{n\times m}$ and $\hat{\Theta}_d\left(t\right) \in R^{N\times m}$ are adaptive time-varying matrices, whose dynamics will be defined later. Substituting (9.12) into (9.11), yields:

$$\begin{aligned} \dot{x} &= A_{ref}\,x + B\,\Lambda\left(\underbrace{\left(\hat{K}_x - K_x\right)}_{\Delta K_x}^T x - \underbrace{\left(\hat{\Theta}_d - \Theta_d\right)}_{\Delta\Theta_d}^T \Phi_d\left(x_p\right)\right) + B_c\,r \\ &= A_{ref}\,x + B\,\Lambda\left(\Delta K_x^T x - \Delta\Theta_d^T\,\Phi_d\left(x_p\right)\right) + B_c\,r \end{aligned} \tag{9.13}$$

Based on (9.13), consider the following *reference model*:

$$\dot{x}_{ref} = A_{ref}\,x_{ref} + B_c\,r \tag{9.14}$$

and let

$$e = x - x_{ref} \tag{9.15}$$

represent the system *tracking error*. Subtracting (9.14) from (9.13) results in the *tracking error dynamics*,

$$\dot{e} = A_{ref} e + B\Lambda\left(\Delta K_x^T x - \Delta\Theta_d^T \Phi_d\left(x_p\right)\right) \tag{9.16}$$

Consider the following Lyapunov function candidate,

$$V\left(e,\, \Delta K_x,\, \Delta\Theta_d\right) = e^T P_{ref}\, e + \text{trace}\left(\Delta K_x^T \Gamma_x^{-1} \Delta K_x \Lambda\right) + \text{trace}\left(\Delta\Theta_d^T \Gamma_{\Theta_d}^{-1} \Delta\Theta_d \Lambda\right) \tag{9.17}$$

where $\Gamma_x = \Gamma_x^T > 0$, $\Gamma_{\Theta_d} = \Gamma_{\Theta_d}^T > 0$ denote constant rates of adaptation, and $P_{ref} = P_{ref}^T > 0$ is the unique symmetric positive definite solution of the algebraic Lyapunov equation,

$$A_{ref}^T P_{ref} + P_{ref} A_{ref} = -Q_{ref} \tag{9.18}$$

with $Q_{ref} = Q_{ref}^T > 0$. Time-differentiating V, along the trajectories of (9.16), gives:

$$\begin{aligned} \dot{V} = &-e^T Q_{ref}\, e + 2e^T P_{ref}\, B\Lambda\left(\Delta K_x^T x - \Delta\Theta_d^T \Phi_d\left(x_p\right)\right) \\ &+2\,\text{trace}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x \Lambda\right) + 2\,\text{trace}\left(\Delta\Theta_d^T \Gamma_{\Theta_d}^{-1} \dot{\hat{\Theta}}_d \Lambda\right) \end{aligned} \tag{9.19}$$

Applying the trace identity (valid for any two co-dimensional vectors a and b),

$$a^T b = trace\left(b\, a^T\right) \tag{9.20}$$

yields:

$$\begin{aligned} \dot{V} = &-e^T Q_{ref}\, e + 2\,\text{trace}\left(\Delta K_x^T \left\{\Gamma_x^{-1} \dot{\hat{K}}_x + x e^T P_{ref}\, B\right\}\Lambda\right) \\ &+2\,\text{trace}\left(\Delta\Theta_d^T \left\{\Gamma_{\Theta_d}^{-1} \dot{\hat{\Theta}}_d - \Phi_d\left(x_p\right) e^T P_{ref}\, B\right\}\Lambda\right) \end{aligned} \tag{9.21}$$

If adaptive laws are selected in the form,

$$\begin{aligned} \dot{\hat{K}}_x &= -\Gamma_x\, x e^T P_{ref} B \\ \dot{\hat{\Theta}}_d &= \Gamma_{\Theta_d} \Phi_d\, e^T P_{ref} B \end{aligned} \tag{9.22}$$

then,

$$V\left(e,\, \Delta K_x,\, \Delta\Theta_d\right) = -e^T Q_{ref}\, e \le 0 \tag{9.23}$$

which in turn, proves UUB of $\left(e,\, \Delta K_x,\, \Delta\Theta_d\right)$. Moreover, one can show that the tracking error signal is square integrable: $e \in L_2$. Since $r \in L_\infty$ then $x_{ref} \in L_\infty$, and consequently, $x \in L_\infty$. Since the ideal (unknown) parameters $\left(K_x,\, \Theta_d\right)$ are constant and their estimation errors $\left(\Delta K_x,\, \Delta\Theta_d\right)$ are bounded, then their estimated values are bounded as well, that is: $\hat{K}_x,\, \hat{\Theta}_d \in L_\infty$. Since components of the regressor vector $\Phi_d\left(x_p\right)$ are locally Lipschitz -continuous functions of x_p, and the latter is bounded, then the components themselves are bounded. Hence, $u \in L_\infty$ and $\dot{x} \in L_\infty$. Thus $\dot{e} \in L_\infty$, which implies that $\ddot{V} \in L_\infty$. Therefore, $\dot{V}$ is a uniformly continuous function of time. Since V is lower bounded, $\dot{V} \le 0$, and $\dot{V}$ is uniformly continuous then V tends to a limit, while its derivative $\dot{V}$ tends to zero, (due to Barbalat' Lemma). Consequently, the tracking error, e tends to zero asymptotically, as $t \to \infty$. Moreover, since the Lyapunov function (9.17) is radially unbounded, then the asymptotic convergence is global, that is the closed-loop tracking error dynamics are globally asymptotically stable.

We have shown that the system state x globally asymptotically tracks the state x_{ref} of the reference model, and therefore the system output $y = C\,x$ globally asymptotically tracks the reference model output $y_{ref} = C\,x_{ref}$. At the same time, the reference model dynamics are chosen such that y_{ref} tracks an external bounded command $r(t)$, with bounded errors. Therefore, y must also track r with bounded errors. The tracking problem is solved. The end result is summarized below.

Theorem 9.1

Consider the uncertain system dynamics in (9.9), operating under the MRAC controller (9.12), with the adaptive laws (9.22). Suppose that the matching relations (9.10) hold. Let the reference model (9.14) be driven by a bounded external command $r(t)$. Then for any symmetric positive definite matrices $\left(\Gamma_x, \Gamma_{\Theta_d}, Q_{ref}\right)$, all signals in the closed-loop system,

$$\begin{cases} \dot{x} \;\; = A\,x + B\,\Lambda\left(\hat{K}_x^T\,x - \hat{\Theta}_d^T\,\Phi_d\left(x_p\right) + \Theta_d^T\,\Phi_d\left(x_p\right)\right) + B_c\,r(t) \\ \dot{x}_{ref} = A_{ref}\,x_{ref} + B_c\,r(t) \\ \dot{\hat{K}}_x = -\Gamma_x \;\; x\left(x - x_{ref}\right)^T P_{ref}\,B \\ \dot{\hat{\Theta}}_d = \;\; \Gamma_{\Theta_d}\,\Phi_d\left(x_p\right)\left(x - x_{ref}\right)^T P_{ref}\,B \end{cases} \tag{9.24}$$

are uniformly ultimately bounded in time, where symmetric positive definite matrix P_{ref} is the unique solution of the algebraic Lyapunov equation (9.18). Moreover, the tracking error signal $e = x - x_{ref}$ is UUB, square integrable, and globally asymptotically stable, $\lim_{t\to\infty}\left\|e(t)\right\| = 0$.

Remark 9.1

As it was shown in the previous section, the MRAC design methodology can be applied to the original system (9.1), and without adding integrated output tracking errors. In this case, the control input is:

$$u = \hat{K}_x^T x + \hat{K}_r^T r - \hat{\Theta}_d^T \Phi_d\left(x_p\right) \tag{9.25}$$

where $\hat{K}_r \in R^{m\times m}$ is the estimated feedforward gain. The system closed-loop dynamics become:

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(\overbrace{\underbrace{\left(\hat{K}_x - K_x\right)^T}_{\Delta K_x} x + \underbrace{\left(\hat{K}_r - K_r\right)^T}_{\Delta K_r} r}^{\overbrace{\left(\Delta K_x^T \quad \Delta K_r^T\right)}^{\Delta K_{\tilde{x}}^T}\overbrace{\begin{pmatrix} x \\ r \end{pmatrix}}^{\tilde{x}}} - \underbrace{\left(\hat{\Theta}_d - \Theta_d\right)^T}_{\Delta\Theta_d} \Theta_d\left(x_p\right)\right) + \underbrace{B\,\Lambda\,K_r^T}_{B_c}\,r \tag{9.26}$$

$$= A_{ref}\,x + B\,\Lambda\left(\Delta K_x^T x - \Delta\Theta_d^T \Phi_d\left(x_p\right)\right) + B_c r$$

where $K_r \in R^{m\times m}$ is the unknown / ideal feedforward gain. The relation $B_c = B\,\Lambda\,K_r^T$ is the additional required matching condition, which guarantees existence of the ideal feedforward gain K_r. The corresponding adaptive laws are:

$$\begin{aligned} \dot{\hat{K}}_x &= -\Gamma_x\, x\, e^T\, P_{ref}\, B \\ \dot{\hat{K}}_r &= -\Gamma_r\, r\, e^T\, P_{ref}\, B \\ \dot{\hat{\Theta}}_d &= \Gamma_{\Theta_d}\, \Phi_d\, e^T\, P_{ref}\, B \end{aligned} \tag{9.27}$$

where $\Gamma_r = \Gamma_r^T > 0$ is the adaptation rate matrix for the feedforward adaptive gain $\hat{K}_r$.

10. MRAC Augmentation of an Optimal Baseline Controller with Integral Action

The adaptive design developed in the previous section can be applied to *augment* a baseline linear controller with (Proportional + Integral) (PI) feedback architecture. The rational for using an augmentation approach (as oppose to all adaptive) stems from the fact that in most realistic applications, a system may already have a baseline controller. This controller would have been designed to operate under nominal conditions (no uncertainties). If adding uncertainties destroys the desired baseline closed-loop performance then one might attempt to recover the desired nominal performance by augmenting the baseline controller with an adaptive element.

We consider a class of multi-input-multi-output (MIMO) nonlinear systems, whose plant dynamics are linearly parameterized, the uncertainties satisfy the matching conditions, and the system state is measurable, (i.e., available on-line as the system output). More specifically, we consider n^{th} order MIMO systems in the form,

$$\dot{x}_p = A_p\, x_p + B_p\, \Lambda\left(u + f\left(x_p\right)\right) \tag{10.1}$$

where n_p and m denote dimensions of the system state x_p and control u, respectively. Also, $x_p \in R^{n_p}$ is the system state, $u \in R^m$ is the control input, $A_p \in R^{n_p\times n_p}$ and $B_p \in R^{n\times m}$ are known, while $\Lambda \in R^{m\times m}$ is unknown. In addition, it is assumed that Λ is diagonal, its elements λ_i are positive, and the pair $\left(A_p, \left(B_p\,\Lambda\right)\right)$ is controllable. The uncertainty Λ is introduced to model control failures.

Moreover, the unknown nonlinear function $f\left(x_p\right): R^{n_p} \to R^m$ represents the *system matched uncertainty*. It is assumed that this function can be written as a linear combination of N known basis functions, with unknown constant coefficients.

$$f\left(x_p\right) = \Theta^T \, \Phi\left(x_p\right) \tag{10.2}$$

In (10.2), $\Theta \in R^{N \times m}$ is the *unknown* constant matrix of ideal parameters, while $\Phi\left(x_p\right) \in R^N$ represents the *known* locally Lipschitz -continuous regressor vector. Thus, we consider MIMO systems of the form:

$$\dot{x}_p = A_p \, x_p + B_p \, \Lambda\left(u + \Theta^T \, \Phi\left(x_p\right)\right) \tag{10.3}$$

with the *controlled* output

$$y = C_p \, x_p + D_p \, \Lambda\left(u + \Theta^T \, \Phi\left(x_p\right)\right) \tag{10.4}$$

Let $y_{cmd}\left(t\right) \in R^m$ denote a bounded command for the system controlled output $y \in R^m$ to follow. This task is to be accomplished using the system control input $u \in R^m$, in the form of a full state feedback. We define output tracking error e_y, its integral e_{yI},

$$\dot{e}_{yI} = e_y = y - y_{cmd} \tag{10.5}$$

and consider the *extended open-loop dynamics*:

$$\underbrace{\begin{pmatrix} \dot{e}_{yI} \\ \dot{x}_p \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0_{m \times m} & C_p \\ 0_{n_p \times m} & A_p \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} e_{yI} \\ x_p \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} D_p \\ B_p \end{pmatrix}}_{B} \Lambda\left(u + f\left(x_p\right)\right) + \underbrace{\begin{pmatrix} -I_{m \times m} \\ 0_{n_p \times m} \end{pmatrix}}_{B_{ref}} y_{cmd} \tag{10.6}$$

or, equivalently

$$\dot{x} = A\,x + B\,\Lambda\left(u + \Theta^T \, \Phi\left(x_p\right)\right) + B_{ref} \, y_{cmd} \tag{10.7}$$

In terms of (10.7), the controlled output y from (10.4) can be written as:

$$y = \underbrace{\begin{pmatrix} 0 & C_p \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} e_{yI} \\ x_p \end{pmatrix}}_{x} + \underbrace{D_p}_{D} \Lambda\left(u + \Theta^T \, \Phi\left(x_p\right)\right) u = C\,x + D\,\Lambda\left(u + \Theta^T \, \Phi\left(x_p\right)\right) u \tag{10.8}$$

The *control problem* of interest is *bounded tracking* in the presence of the system constant parametric uncertainties Λ and Θ. Specifically, one needs to design the control input u, so that the system controlled output y tracks any bounded time-varying command y_{cmd}, with bounded tracking errors, while the rest of the signals in the corresponding closed-loop dynamics remain bounded.

We begin with the design of a baseline linear controller. Setting $\Lambda = I_{m \times m}$, $\Theta = 0_{N \times m}$ in (10.7), results in the linear time invariant (LTI) *baseline* dynamics.

$$\begin{aligned} \dot{x} &= A\,x + B\,u + B_{ref} \, y_{cmd} \\ y &= C\,x + D\,u \end{aligned} \tag{10.9}$$

Assuming *constant* command y_{cmd}, we can use the Linear Quadratic Regulator (LQR) method, with Proportional + Integral (PI) feedback connections, to design a baseline *optimal* control law. This controller represents the so-called LQR PI *servomechanism*. Its design is outlined below.

We first calculate an *optimal* stabilizing controller for:

$$\dot{z} = A z + B v \tag{10.10}$$

where

$$z = \dot{x} = \begin{pmatrix} \dot{e}_{yI} \\ \dot{x}_p \end{pmatrix}, \quad v = \dot{u} \tag{10.11}$$

and the control input is designed to minimize the Linear Quadratic cost:

$$J(v) = \int_0^\infty \left(z^T Q z + v^T R v \right) dt \to \min_v \tag{10.12}$$

with the appropriately selected symmetric positive definite matrices Q and R. It is well-known that the corresponding optimal LQR solution is given in *feedback* form:

$$\dot{u} = v = -\underbrace{R^{-1} B^T P}_{K_x^T} z = -\begin{pmatrix} K_I & K_P \end{pmatrix} \begin{pmatrix} \dot{e}_{yI} \\ \dot{x}_p \end{pmatrix} \tag{10.13}$$

In (10.13), P is the unique symmetric positive semi-definite solution of the algebraic Riccati equation,

$$A^T P + P A + Q - P B R^{-1} B^T P = 0 \tag{10.14}$$

which is solved using appropriately chosen $Q = Q^T \geq 0$. Integrating (10.13), yields the *baseline LQR PI controller*,

$$u_{bl} = -K_x^T x = -K_I e_{yI} - K_P x = K_I \frac{(y_{cmd} - y)}{s} - K_P x_p \tag{10.15}$$

where the optimal gain matrix

$$K_x^T = \begin{pmatrix} K_I & K_P \end{pmatrix} \tag{10.16}$$

is partitioned into the integral gain K_I and the proportional gain K_P. The corresponding control block-diagram is shown in Figure 11.1.

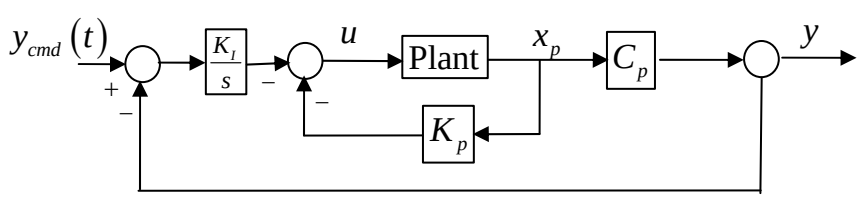

Figure 11.1. Baseline / Servomechanism LQR PI Control Block-Diagram

In the presence of the system uncertainties Λ and Θ, the baseline tracking performance will often deteriorate. In order to restore the desired baseline behavior, an adaptive augmentation can be added. This process consists of: a) A reference model definition, b) Tracking dynamics formulation, and b) The design of adaptive laws.

We define the reference model to represent the baseline closed-loop system dynamics, which are obtained by substituting the baseline controller (10.15) into the LTI system (10.9). The resulting *reference model dynamics* are of the form:

$$\begin{aligned} \dot{x}_{ref} &= A_{ref}\, x_{ref} + B_{ref}\, y_{cmd} \\ y_{ref} &= C_{ref}\, x_{ref} \end{aligned} \tag{10.17}$$

where

$$\begin{aligned} A_{ref} &= A - B\, K_x^T \\ C_{ref} &= C - D\, K_x^T \end{aligned} \tag{10.18}$$

and A_{ref} is Hurwitz by design. It is easy to see that the DC gain from the command y_{cmd} to the reference output y_{ref} is unity, (**Homework**: Prove this statement).

Total control input is defined as the sum of the baseline LQR PI component (10.15) and its adaptive augmentation u_{ad},

$$u = \underbrace{-K_x^T\, x}_{u_{bl}} + u_{ad} = u_{bl} + u_{ad} \tag{10.19}$$

Substituting (10.19) into the original system dynamics (10.7), gives:

$$\begin{aligned} \dot{x} &= A_{ref}\, x + B\,\Lambda \left(u_{ad} + \underbrace{\overbrace{\left(I_{m\times m} - \Lambda^{-1}\right)}^{K_u^T} u_{bl} + \Theta^T\, \Phi\left(x_p\right)}_{\bar{\Theta}^T\, \bar{\Phi}\left(u_{bl}, x_p\right)} \right) + B_{ref}\, y_{cmd} \\ y &= C_{ref}\, x + D\,\Lambda\left(u_{ad} + \bar{\Theta}^T\, \bar{\Phi}\left(u_{bl}, x_p\right)\right) \end{aligned} \tag{10.20}$$

or equivalently,

$$\begin{aligned} \dot{x} &= A_{ref}\, x + B\,\Lambda\left(u_{ad} + \bar{\Theta}^T\, \bar{\Phi}\left(u_{bl}, x_p\right)\right) + B_{ref}\, y_{cmd} \\ y &= C_{ref}\, x + D\,\Lambda\left(u_{ad} + \bar{\Theta}^T\, \bar{\Phi}\left(u_{bl}, x_p\right)\right) \end{aligned} \tag{10.21}$$

with the redefined regressor vector

$$\bar{\Phi}\left(u_{bl}, x_p\right) = \left(u_{bl}^T \quad \Phi^T\left(x_p\right)\right)^T \tag{10.22}$$

and with the extended matrix of unknown / ideal parameters.

$$\bar{\Theta} = \left(K_u^T \quad \Theta^T\right)^T \tag{10.23}$$

The adaptive component u_{ad} is chosen to dominate / approximate the system matched uncertainty $\bar{\Theta}^T\, \bar{\Phi}\left(x\right)$.

$$u_{ad} = -\hat{\bar{\Theta}}^T\, \bar{\Phi}\left(u_{bl}, x_p\right) \tag{10.24}$$

where $\hat{\bar{\Theta}} \in R^{(n+N)\times m}$ is the matrix of adaptive parameters. Substituting (10.24) into (10.21), results in:

$$\dot{x} = A_{ref}\ x - B\,\Lambda \underbrace{\left(\hat{\bar{\Theta}} - \bar{\Theta}\right)^T}_{\Delta\bar{\Theta}} \bar{\Phi} + B_{ref}\ y_{cmd}$$
$$y = C_{ref}\ x - D\,\Lambda\,\Delta\bar{\Theta}^T\ \bar{\Phi} \tag{10.25}$$

where,

$$\Delta\bar{\Theta} = \hat{\bar{\Theta}} - \bar{\Theta} \tag{10.26}$$

is the matrix of *parameter estimation errors*. Introduce the *tracking error*:

$$e = x - x_{ref} \tag{10.27}$$

Then the *tracking error dynamics* can be calculated by subtracting the reference system dynamics (10.17) from the open-loop system dynamics (10.25).

$$\dot{e} = A_{ref}\ e - B\,\Lambda\,\Delta\bar{\Theta}^T\bar{\Phi} \tag{10.28}$$

Consider the Lyapunov function candidate:

$$V\left(e,\ \Delta\bar{\Theta}\right) = e^T\ P_{ref}\ e + \text{trace}\left(\Delta\bar{\Theta}^T\ \Gamma_{\bar{\Theta}}^{-1}\ \Delta\bar{\Theta}\,\Lambda\right) \tag{10.29}$$

where $\Gamma_{\bar{\Theta}} = \Gamma_{\bar{\Theta}}^T > 0$ denotes constant rates of adaptation, and $P_{ref} = P_{ref}^T > 0$ is the unique symmetric positive definite solution of the algebraic Lyapunov equation,

$$A_{ref}^T P_{ref} + P_{ref} A_{ref} = -Q_{ref} \tag{10.30}$$

with $Q_{ref} = Q_{ref}^T > 0$. Time-differentiating V, along the trajectories of (10.28), gives:

$$\dot{V} = -e^T\ Q_{ref}\ e - 2\,e^T\ P_{ref}\ B\,\Lambda\,\Delta\bar{\Theta}^T\ \bar{\Phi} + 2\,\text{trace}\left(\Delta\bar{\Theta}^T\ \Gamma_{\bar{\Theta}}^{-1}\ \dot{\hat{\bar{\Theta}}}\,\Lambda\right) \tag{10.31}$$

Applying the trace identity,

$$a^T\ b = trace\left(b\,a^T\right) \tag{10.32}$$

yields:

$$\dot{V} = -e^T\ Q_{ref}\ e + 2\,\text{trace}\left(\Delta\bar{\Theta}^T\left\{\Gamma_{\bar{\Theta}}^{-1}\ \dot{\hat{\bar{\Theta}}} - \bar{\Phi}\,e^T\ P_{ref}\ B\right\}\Lambda\right) \tag{10.33}$$

If adaptive laws are selected in the form,

$$\dot{\hat{\bar{\Theta}}} = \Gamma_{\bar{\Theta}}\ \bar{\Phi}\left(u_{bl}, x_p\right)\ e^T\ P_{ref}\ B \tag{10.34}$$

then,

$$V\left(e,\ \Delta\bar{\Theta}\right) = -e^T\ Q_{ref}\ e \le 0 \tag{10.35}$$

which in turn, proves UUB of $\left(e,\ \Delta\bar{\Theta}\right)$.

Moreover, it follows from (10.35) that the tracking error signal is square integrable, $e \in L_2$. Since $y_{cmd} \in L_\infty$ then $x_{ref} \in L_\infty$, and consequently, $x \in L_\infty$ and $\left(u_{bl},\ x_p\right) \in L_\infty$. Since the ideal (unknown) matrix of parameters $\bar{\Theta}$ is constant and the estimation errors $\Delta\bar{\Theta}$ are bounded, then their estimated values are bounded as well, that is: $\hat{\bar{\Theta}} \in L_\infty$. Since components of the regressor vector $\bar{\Phi}\left(u_{bl}, x_p\right)$ are locally Lipschitz -continuous, and $\left(u_{bl},\ x_p\right) \in L_\infty$ then the regressor components are bounded. Hence, $u \in L_\infty$ and $\dot{x} \in L_\infty$.

Thus $\dot{e} \in L_\infty$, which implies that $\ddot{V} \in L_\infty$. Therefore, $\dot{V}$ is a uniformly continuous function of time. Since V is lower bounded, $\dot{V} \le 0$, and $\dot{V}$ is uniformly continuous then V tends to a limit, while its derivative $\dot{V}$ tends to zero, (due to Barbalat' Lemma). Consequently, the tracking error e tends to zero asymptotically, as $t \to \infty$. Moreover, since the Lyapunov function (10.29) is radially unbounded, then the asymptotic convergence is global, that is the closed-loop tracking error dynamics (10.28) are globally asymptotically stable. Consequently, $\lim_{t\to\infty} \left\| \Delta\bar{\Theta}^T(t)\, \bar{\Phi}\left(u_{bl}(t),\, x(t)\right) \right\| = 0$ and,

$$y = C\,x - D\,\Lambda\,\Delta\bar{\Theta}^T\,\bar{\Phi} \to C_{ref}\,x \to C_{ref}\,x_{ref} = y_{ref} \tag{10.36}$$

that is for any bounded command y_{cmd}, the closed-loop system output from (10.25) globally asymptotically tracks the reference model output from (10.17), as $t \to \infty$. At the same time, the reference model dynamics (10.17) are chosen such that y_{ref} tracks an external bounded command $y_{cmd}(t)$, with bounded errors. Therefore, y must also track y_{cmd} with bounded errors. The tracking problem is solved.

Remark 10.1

The adaptive laws (10.34) can be written in terms of the system original parameters. Let's partition the adaptive rate matrix,

$$\Gamma_{\bar{\Theta}} = \begin{pmatrix} \Gamma_u & 0_{n\times m} \\ 0_{N\times m} & \Gamma_\Theta \end{pmatrix} \tag{10.37}$$

where $\left(\Gamma_u, \Gamma_\Theta\right)$ denote the adaptive rates for uncertainties that correspond to x and $\Phi\left(x_p\right)$. Using (10.22), (10.23), (10.37), the adaptive laws (10.34) can now be written as:

$$\begin{aligned} \dot{\hat{K}}_u &= \Gamma_u\, u_{bl}\, e^T\, P_{ref}\, B \\ \dot{\hat{\Theta}} &= \Gamma_\Theta\, \Phi\left(x_p\right) e^T\, P_{ref}\, B \end{aligned} \tag{10.38}$$

Also, the (LQR PI Baseline + Adaptive) total control input (10.19) becomes,

$$u = u_{bl} + u_{ad} = \underbrace{\boxed{-K_x^T\, x}}_{u_{bl}\text{ = Baseline}} + \underbrace{\boxed{\left[-\hat{K}_u^T\, u_{bl} - \hat{\Theta}^T\, \Phi\left(x_p\right)\right]}}_{u_{ad}\text{ = Adaptive Augmentation}} \tag{10.39}$$

or, equivalently

$$\begin{aligned} u &= \left(I_{m\times m} - \hat{K}_u^T\right) u_{bl} - \hat{\Theta}^T\, \Phi\left(x_p\right) = -\left(I_{m\times m} - \hat{K}_u^T\right) K_x^T\, x - \hat{\Theta}^T\, \Phi\left(x_p\right) \\ &= \left(I_{m\times m} - \hat{K}_u^T\right)\left(K_I\, \frac{\left(y_{cmd} - y\right)}{s} - K_P\, x_p \right) - \hat{\Theta}^T\, \Phi\left(x_p\right) \end{aligned} \tag{10.40}$$

Note that by the design, this controller *does not* have a feedforward component. Also note that in the adaptive laws (10.38), the adaptive parameter initial values can be set to zero.