Lecture 2

4. LaSalle's Invariance Principle

We begin with a motivating example.

Example 4.1 (nonlinear pendulum dynamics with friction)

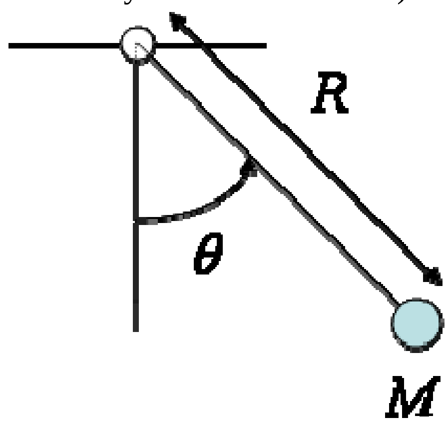

Figure 4.1: Pendulum

Dynamics of a pendulum with friction can be written as:

$$M\,R^2\,\ddot{\theta}+k\,\dot{\theta}+M\,g\,R\sin(\theta)=0 \tag{4.1}$$

or, equivalently in state space form:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - b\,x_2\end{aligned} \tag{4.2}$$

where $x_1=\theta$, $x_2=\dot{\theta}$, $a=\dfrac{g}{R}$, and $b=\dfrac{k}{M\,R^2}$. We study stability of the origin $x_e=0$. Note that the latter is equivalent to studying stability of all the equilibrium points in the form: $x_e=\begin{pmatrix}2\pi l & 0\end{pmatrix}^T$, $l=0,\pm1,\pm2,\ldots$ Consider the total energy of the pendulum as a Lyapunov function candidate.

$$V(x)=\underbrace{\int_0^{x_1} a\sin y\,dy}_{\text{Potential}}+\underbrace{\frac{x_2^2}{2}}_{\text{Kinetic}}=a(1-\cos x_1)+\frac{x_2^2}{2} \tag{4.3}$$

It is clear that $V(x)$ is a positive definite function, (locally, around the origin). Its time derivative along the system trajectories is:

$$\dot{V}(x)=a\sin x_1\,\dot{x}_1+x_2\,\dot{x}_2=-b\,x_2^2\le 0 \tag{4.4}$$

So, the time derivative is negative <u>*semidefinite*</u>. It is not strictly negative definite because $\dot{V}(x)=0$ for $x_2=0$, irrespective of the value of x_1. Therefore, we can only conclude that the origin is a stable equilibrium, but not necessarily asymptotically stable.

However, using the phase portrait of the pendulum equation (or just common sense), we expect the origin to be *asymptotically* stable. The Lyapunov energy function argument fails to show it.

On the other hand, we notice that for the system to maintain $\dot{V}(x)=0$ condition, the trajectory must be confined to the line $x_2=0$. Using the system dynamics (4.2) yields:

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow \sin x_1 \equiv 0 \Rightarrow x_1 \equiv 0$$

Hence on the segment $-\pi < x_1 < \pi$ of the line $x_2=0$ the system can *maintain* the $\dot{V}(x)=0$ condition only at the origin $x=0$. Therefore, $V(x(t))$ must decrease to toward 0 and, consequently, $x(t)\to 0$ as $t\to\infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The forgoing argument shows that if in a domain about the origin we can find a Lyapunov function whose derivative along the system trajectories is negative semidefinite, and we can establish that no trajectory can stay identically at points where $\dot{V}(x)=0$, except at the origin, then the origin is asymptotically stable. This argument follows from the *LaSalle's Invariance Principle* which is applicable to autonomous systems of the form

$$\dot{x} = f(x), \quad f(0)=0 \tag{4.5}$$

Definition 4.1

A set $M \subset R^n$ is said to be

- an *invariant set* with respect to (4.5) if: $x(0)\in M \Rightarrow x(t)\in M, \quad \forall t \in R$
- a *positively invariant set* with respect to if: $x(0)\in M \Rightarrow x(t)\in M, \quad \forall t \geq 0$

Theorem 4.1 (LaSalle's Theorem)

Let $\Omega \subset D \subset R^n$ be a *compact positively invariant set* with respect to the system dynamics (4.5). Let $V: D \to R$ be a continuously differentiable function such that $\dot{V}(x(t)) \leq 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $\dot{V}(x)=0$. Let $M \subset E$ be the largest invariant set in E. Then every solution starting in Ω approaches M as $t\to\infty$, that is

$$\lim_{t\to\infty}\left(\underbrace{\inf_{z\in M}\left\|x(t)-z\right\|}_{\mathrm{dist}(x(t),M)}\right)=0$$

Notice that the inclusion of the sets in the LaSalle's theorem is:

$$M \subset E \subset \Omega \subset D \subset R^n$$

In fact, a formal proof of the theorem reveals that all trajectories $x(t)$ are bounded and approach a ***positive limit set*** $L^{+} \subset M$ as $t \to \infty$. The latter may contain asymptotically stable equilibriums and stable limit cycles.

Remark 4.1

Unlike Lyapunov theorems, LaSalle's theorem ***does not*** require the function $V(x)$ to be positive definite.

Most often, our interest will be to show that $x(t) \to 0$ as $t \to \infty$. For that we will need to establish that the largest invariant set in E is the origin, that is: $M = \{0\}$. This is done by showing that no solution can stay identically in E other than the trivial solution $x(t) \equiv 0$.

Theorem 4.1 (Barbashin-Krasovskii theorem)

Let $x = 0$ be an equilibrium point for (4.5). Let $V : D \to R$ be a *continuously differentiable positive definite* function on a domain $D \subset R^{n}$ containing the origin, such that $\dot{V}(x(t)) \le 0$ in D. Let $S = \{x \in D : \dot{V}(x) = 0\}$ and suppose that no other solution can stay in S, other than the trivial solution $x(t) \equiv 0$. Then the origin is *locally asymptotically stable*. If, in addition, $V(x)$ is *radially unbounded* then the origin is *globally asymptotically stable*.

Note that if $\dot{V}(x)$ is negative definite then $S = \{0\}$ and the above theorem coincides with the Lyapunov 2nd theorem. Also note that the LaSalle's invariant set theorems are applicable to autonomous system only.

Example 4.2

Consider the 1st order system

$$\dot{x} = a\,x + u$$

together with its *adaptive control law*

$$u = -\hat{k}(t)\,x$$

The dynamics of the adaptive gain $\hat{k}(t)$ is

$$\dot{\hat{k}} = \gamma\,x^{2}$$

where $\gamma > 0$ is called the adaptation rate. Then the closed-loop system becomes:

$$\begin{cases} \dot{x} = -\left(\hat{k}(t) - a\right)x \\ \dot{\hat{k}} = \gamma\,x^{2} \end{cases}$$

The line $x = 0$ represents the system equilibrium set. We want to show that the trajectories approach this equilibrium set, as $t \to \infty$, which means that the adaptive controller regulates $x(t)$ to zero in the presence of constant uncertainty in a. Consider the Lyapunov function candidate

$$V\left(x,\hat{k}\right)=\frac{1}{2}x^2+\frac{1}{2\gamma}\left(\hat{k}-b\right)$$

where $b>a$. The time derivative of V along the trajectories of the system is given by

$$\dot{V}\left(x,\hat{k}\right)=x\,\dot{x}+\frac{1}{\gamma}\left(\hat{k}-b\right)\dot{\hat{k}}=-x^2\left(\hat{k}-a\right)+\left(\hat{k}-b\right)x^2=-x^2\left(b-a\right)\leq 0$$

Since $V\left(x,\hat{k}\right)$ is positive definite and radially unbounded function, whose derivative $\dot{V}\left(x,\hat{k}\right)\leq 0$ is semi-negative, then $\Omega_c=\left\{\left(x,\hat{k}\right)\in R^2 : V\left(x,\hat{k}\right)\leq c\right\}$ is compact, positively invariant set. Thus, taking $\Omega=\Omega_c$, all the conditions of LaSalle's Theorem are satisfied. The set E is given by $E=\left\{\left(x,\hat{k}\right)\in\Omega_c : x=0\right\}$. Since any point on the line $x=0$ is an equilibrium point, E is an invariant set. Therefore, in this example $M=E$. From LaSalle's Theorem we conclude that every trajectory starting in Ω_c approaches E, as $t\to\infty$, that is $x(t)\to 0$ as $t\to\infty$. Moreover, since $V\left(x,\hat{k}\right)$ is radially unbounded, the conclusion is <u>*global*</u>, that is it holds for <u>*all*</u> initial conditions $\left(x(0),\hat{k}(0)\right)$ because the constant c in the definition of Ω_c can be chosen large enough that $\left(x(0),\hat{k}(0)\right)\in\Omega_c$.

Homework:

- Simulate the closed-loop system from Example 4.2.
- Test different initial conditions $\left(x(0),\hat{k}(0)\right)$.
- Run simulations of the system while increasing the rate of adaptation $\gamma>0$ until high frequency oscillations and / or system departure occurs. Try to quantify maximum allowable $\gamma_{\max}$ as a function of the initial conditions.
- Run simulations of the system while increasing the control time delay $\tau\geq 0$, that is using control in the form $u(t)=-\hat{k}(t-\tau)x(t-\tau)$. Try to quantify maximum allowable time delay $\tau_{\max}$, (as a function of the initial conditions and rate of adaptation), before the system starts to oscillate or departs.

5. Boundedness and Ultimate Boundedness

Consider the nonautonomous system

$$\dot{x}=f(t,x) \tag{5.1}$$

where $f:[0,\infty)\times D\to R^n$ is piecewise continuous in t, locally Lipschitz in x on $[0,\infty)\times D$, and $D\subset R^n$ is a domain that contains the origin $x=0$. If the origin is the equilibrium point for (5.1) then by definition: $f(t,0)=0,\quad\forall t\geq 0$. On the other hand, even if there is no equilibrium at the origin, Lyapunov analysis can still be used to show boundedness of the system trajectories. We begin with a motivating example.

Example 5.1

Consider the IVP with nonautonomous scalar dynamics

$$\dot{x} = -x + \delta \sin t$$
$$x(t_0) = a > \delta > 0 \tag{5.2}$$

The system has no equilibrium points. The IVP explicit solution can be easily found and shown to be bounded for all $t \geq t_0$, uniformly in t_0, that is with a bound b independent of t_0. In this case, the solution is said to be *uniformly ultimately bounded* (UUB), and b is called the ultimate bound, (Homework: Prove this statement).

The UUB property of (5.2) can be established via Lyapunov analysis and without using explicit solutions of the state equation. Starting with $V(x) = \frac{x^2}{2}$, we calculate the time derivative of V along the system trajectories.

$$\dot{V}(x) = x\,\dot{x} = x(-x + \delta \sin t) = -x^2 + \delta\, x \sin t \leq -x^2 + \delta |x| = -|x|(|x| - \delta)$$

It immediately follows that

$$\dot{V}(x) < 0, \quad \forall |x| > \delta$$

In other words, the time derivative of V is negative outside the set $B_\delta = \{|x| \leq \delta\}$, or equivalently, all solutions that start outside of B_δ will reenter the set within a finite time, and will remain their afterward. Formally, it can be stated as follows. Choose $c > \frac{\delta^2}{2}$. Since $\dot{V}$ is negative on the set boundary then all solutions starting in the set

$$B_c = \underbrace{\{V(x) \leq c\}}_{|x| \leq \sqrt{2c}} \supset B_\delta$$

will remain therein for all future time. Hence the solutions are *uniformly bounded*. Moreover, an ultimate bound of these solutions can also be found. Choose ε such that

$$\frac{\delta^2}{2} < \varepsilon < c$$

Then $\dot{V}$ is negative in the annulus set $\{\varepsilon \leq V(x) \leq c\}$, which implies that in this set $V(x(t))$ will decrease monotonically in time until the solution enters the set $\{V(x) \leq \varepsilon\}$. From that time on, the solution cannot leave the set because $\dot{V}$ is negative on its boundary $V(x) = \varepsilon$. Since $V(x) = \frac{x^2}{2}$, we can conclude that these solutions are UUB with the *ultimate bound* $|x| \leq \sqrt{2\varepsilon}$.

Definition 5.1

The solutions of (5.1) are

- *Uniformly Bounded* if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0,\ c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0 \tag{5.3}$$

- *Globally Uniformly Bounded* if (5.3) holds for arbitrarily large a.
- *Uniformly Ultimately Bounded* with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0,\ c)$, there is $T = T(a,b)$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall t \geq t_0 + T \tag{5.4}$$

- *Globally Uniformly Ultimately Bounded* if (5.4) holds for arbitrarily large a.

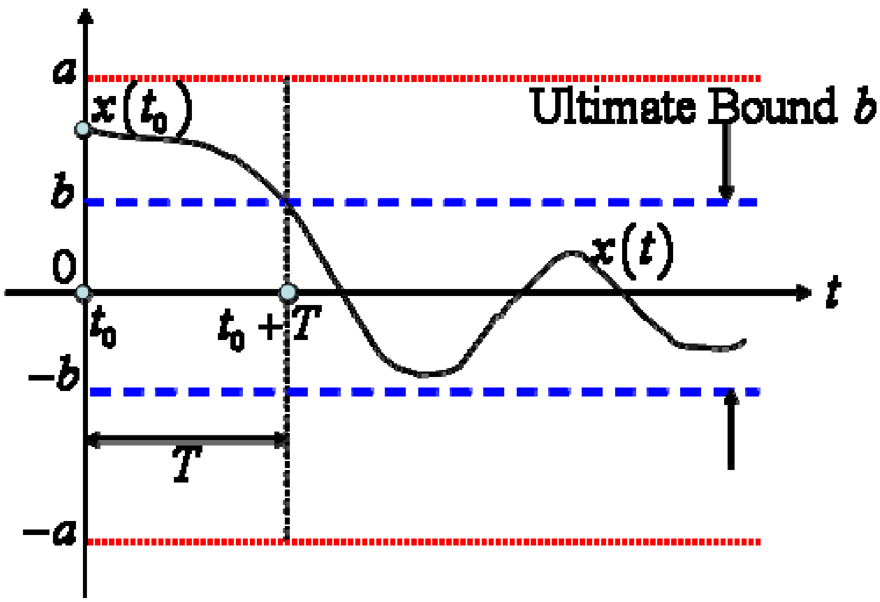

Figure 5.1: UUB Concept

In the definition above, the term *uniform* indicates that the bound b does not depend on t_0. The term *ultimate* means that boundedness holds after the lapse of a certain time T. The constant c defines a neighborhood of the origin, independent of t_0, such that all trajectories starting in the neighborhood will remain bounded in time. If c can be chosen arbitrarily large then the UUB notion becomes global.

Basically, UUB can be considered as a "milder" form of stability in the sense of Lyapunov (SISL). A comparison summary between SISL and UUB concepts is given below.

- SISL is defined with respect to an equilibrium, while UUB is not.
- Asymptotic SISL is a strong property that is very difficult to achieve in practical dynamical systems.

- SISL requires the ability to keep the state arbitrarily close to the system equilibrium by starting sufficiently close to it. This is still too strong a requirement for practical systems operating in the presence of unknown disturbances.
- The main difference between UUB and SISL is that the UUB bound b cannot be made arbitrarily small by starting closer to the equilibrium or the origin. In practical systems, the bound b depends on disturbances and system uncertainties.

To demonstrate how Lyapunov analysis can be used to study UUB, consider a continuously differentiable positive definite function $V(x)$, choose $0 < \varepsilon < c$, and suppose that the sets $\Omega_\varepsilon = \{V(x) \le \varepsilon\}$ and $\Omega_c = \{V(x) \le c\}$ are compact. Let

$$\Lambda = \{\varepsilon \le V(x) \le c\} = \Omega_c - \Omega_\varepsilon$$

and suppose that it is known that the time derivative of $V(x(t))$ along the trajectories of the nonautonomous dynamical system (5.1) is negative definite inside Λ, that is

$$\dot{V}(x(t)) \le -W(x(t)) < 0, \quad \forall x \in \Lambda, \quad \forall t \ge t_0$$

where $W(x(t))$ is a continuous positive definite function. Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in the direction of decreasing $V(x(t))$. It can be shown that in the set Λ the trajectory behaves as if the origin was uniformly asymptotically stable, (which it does not have to be in this case). Consequently, the function $V(x(t))$ will continue decreasing until the trajectory enters the set Ω_ε *in finite time* and stays there for all future time. Hence, the solutions of (5.1) are UUB with the ultimate bound $b = \max_{x \in \Omega_\varepsilon} \|x\|$. A sketch of the sets Λ, Ω_c, Ω_ε is shown in Figure 5.2.

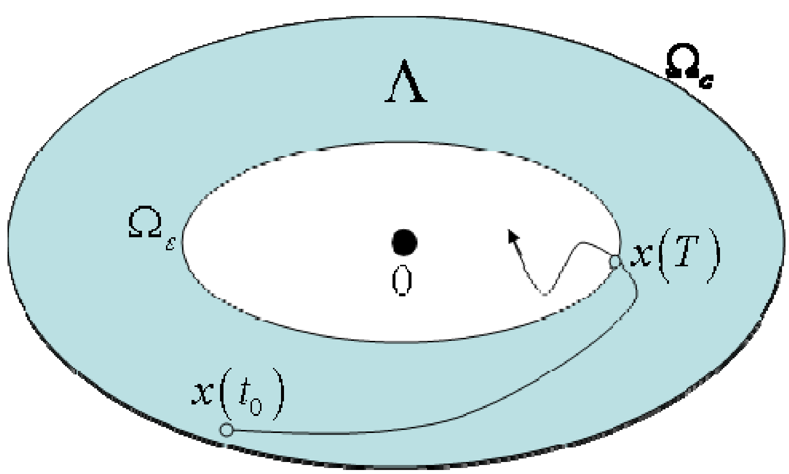

Figure 5.2: UUB by Lyapunov Analysis

In many problems, the relation $\dot{V}(t,x) \le -W(x)$ is derived and shown to be valid on a domain, which is specified in terms of a Euclidian norm $\|x\|$. In such cases, UUB analysis involves finding the corresponding domains of attraction and an ultimate bound.

This analysis will be performed next. Let $\Omega \subset R^n$ denote a bounded domain and suppose that the system dynamics are:

$$\dot{x} = A\,x + B\,\frac{\varepsilon(t,x)}{2} \tag{5.5}$$

where A is Hurwitz. Also suppose that $\varepsilon(t,x) \in R^m$ is a bounded function on Ω. Let $Q = Q^T > 0$ and consider

$$V(x) = x^T\,P\,x \tag{5.6}$$

where $P = P^T > 0$ is the unique positive definite symmetric solution of the algebraic Lyapunov equation.

$$P\,A + A^T\,P = -Q \tag{5.7}$$

Then the time derivative of V evaluated along the system (5.1) trajectories satisfies the following relation:

$$\dot{V}(x) = -x^T\,Q\,x + x^T\,P\,B\,\varepsilon(t,x), \quad \forall\, x \in \Omega, \quad \forall\, t \geq t_0 \tag{5.8}$$

Suppose that $R > 0$ is chosen such that the sphere S_R is inside the domain Ω, that is:

$$S_R = \left\{ \|x\| \leq R \right\} \subset \Omega \tag{5.9}$$

Also suppose that $\left|\varepsilon(t,x)\right| \leq \varepsilon_{\max} < \infty$ for all $x \in S_R$, uniformly in t. Then one can formally prove that all the solutions $x(t)$ of (5.1) that start in a subset of S_R are UUB.

In order to prove the UUB property, we assume that $\|x\| = \sqrt{x^T\,x}$. Then for any symmetric positive definite matrix P and for all vectors x,

$$\lambda_{\min}(P)\|x\|^2 \leq x^T\,P\,x \leq \lambda_{\max}(P)\|x\|^2 \tag{5.10}$$

where $\lambda_{\min}(P),\, \lambda_{\max}(P)$ denote the smallest and the largest eigenvalues of P, respectively.

An upper bound for $\dot{V}$ in (5.8) can be calculated, for $x \in \Omega$.

$$\dot{V}(x) \leq -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|P\,B\|\varepsilon_{\max} = -\|x\|\left(\lambda_{\min}(Q)\|x\| - \|P\,B\|\varepsilon_{\max}\right) \tag{5.11}$$

Define a sphere.

$$S_r = \left\{ \|x\| \leq r = \frac{\|P\,B\|\varepsilon_{\max}}{\lambda_{\min}(Q)} \right\} \tag{5.12}$$

It follows from (5.11) that

$$\dot{V}(x) < 0, \quad \forall\, x \in \Lambda = \left\{ r \leq \|x\| \leq R \right\} = S_R - S_r \tag{5.13}$$

Let $b_r = \lambda_{\max}(P)\,r^2$ and let $\Omega_{b_r} = \left\{ V(x) \leq b_r \right\}$. Then $S_r \subset \Omega_{b_r}$. In fact, if $x \in S_r$ then using the right hand side of (5.10) yields:

$$x^T\,P\,x \leq \lambda_{\max}(P)\|x\|^2 \leq \lambda_{\max}(P)\,r^2 = b_r \tag{5.14}$$

Hence, $x \in \Omega_{b_r}$ and the inclusion $S_r \subset \Omega_{b_r}$ is proven.

Let $b_R = \lambda_{\min}(P) R^2$ and define $\Omega_{b_R} = \{V(x) \le b_R\}$. Then $\Omega_{b_R} \subset S_R$. In fact, if $x \in \Omega_{b_R}$ then using the left hand side of (5.10), yields:

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x = V(x) \le b_R = \lambda_{\min}(P) R^2 \tag{5.15}$$

Hence, $\|x\| \le R$, that is $x \in S_R$, and the inclusion $\Omega_{b_R} \subset S_R$ is proven.

Next, we need to ensure that $b_r < b_R$, that is:

$$b_r = \lambda_{\max}(P) r^2 < \lambda_{\min}(P) R^2 = B_R \tag{5.16}$$

or, equivalently:

$$\frac{r}{R} < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \tag{5.17}$$

The above inequality can be viewed as a restriction placed on the eigenvalues of P and the constants r and R. This relation ensures the desired set inclusions:

$$S_r \subset \Omega_{b_r} \subset \Omega_{b_R} \subset S_R \tag{5.18}$$

Graphical representation of the four sets is given in Figure 5.2.

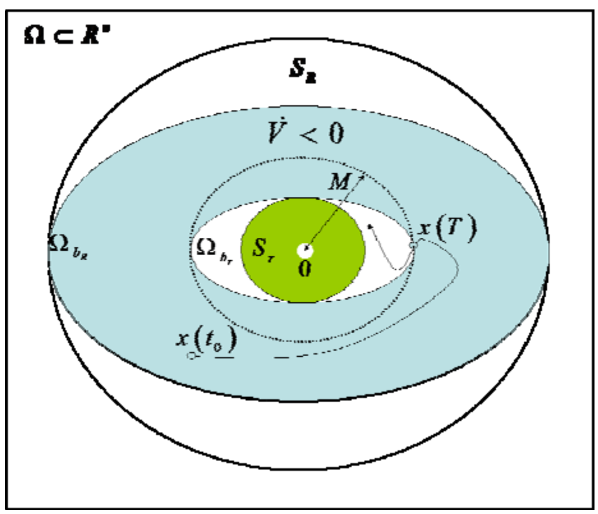

Figure 5.2: Representation of the sets $S_r \subset \Omega_{b_r} \subset \Omega_{b_R} \subset S_R$ and the Ultimate Bound M

Next, we show that all solutions starting in Ω_{b_R} will enter Ω_{b_r} and remain there afterwards. If $x(t_0) \in \Omega_{b_r}$ then since $\dot{V} < 0$ in $\Lambda = \Omega_{b_R} - \Omega_{b_r}$, $V(x(t))$ is a decreasing function of time outside of Ω_{b_r}. Therefore, solutions that start in Ω_{b_r} will remain there. Suppose that $x(t_0) \in \Lambda$. Inequality (5.11) implies:

$$\begin{aligned}\dot{V}(x) &\le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|PB\|\varepsilon_{\max} \\ &= -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\underbrace{\lambda_{\max}(P)\|x\|^2}_{\ge V(x)} + \frac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}\underbrace{\sqrt{\lambda_{\min}(P)}\|x\|}_{\le\sqrt{V(x)}} \\ &\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(x) + \frac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}\sqrt{V(x)}\end{aligned} \tag{5.19}$$

Thus, $V(x(t)) \ge 0$ satisfies the following differential inequality, (as a function of time):

$$\dot{V}(x) \le -aV(x) + g\sqrt{V(x)} \tag{5.20}$$

where $a = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and $g = \frac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}$ are positive constants. Let $W(x) = \sqrt{V(x)}$. Then (5.20) is equivalent to

$$\dot{W}(x) \le -\frac{a}{2}W(x) + \frac{g}{2} \tag{5.21}$$

Define

$$Z(x) = \dot{W}(x) + \frac{a}{2}W(x) \tag{5.22}$$

Then because of (5.21),

$$Z(x(t)) \le \frac{g}{2}, \quad \forall t \ge t_0 \tag{5.23}$$

Solving (5.22) for W yields,

$$W(x(t)) = W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} Z(x(\tau))\,d\tau \tag{5.24}$$

and, therefore:

$$\begin{aligned}W(x(t)) &\le W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} \left|Z(x(\tau))\right| d\tau \\ &\le W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \frac{g}{2}\int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)}\,d\tau = W(x(t_0))e^{-\frac{a}{2}(t-t_0)} + \frac{g}{a}\left[1 - e^{-\frac{a}{2}(t-t_0)}\right] \\ &= \frac{g}{a} + \underbrace{e^{-\frac{a}{2}(t-t_0)}\left[W(x(t_0)) - \frac{g}{a}\right]}_{\mathrm{o}(1)} = \frac{g}{a} + \mathrm{o}(1)\end{aligned} \tag{5.25}$$

Consequently, as $t \to \infty$:

$$\sqrt{x^T(t)Px(t)} \le \frac{g}{a} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\underbrace{\left(\frac{\|PB\|\varepsilon_{\max}}{\lambda_{\min}(Q)}\right)}_{r} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}r + \mathrm{o}(1) \tag{5.26}$$

Choose $\delta > 0$. Then it is easy to see that there exists T independent of t_0 such that $\mathrm{o}(1) \le \delta$ and , consequently

$$\sqrt{x^T(t)\,P\,x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r + \delta, \quad \forall\, t \ge T + t_0, \quad \forall\, x \in \Omega_{b_R} \tag{5.27}$$

Since the above relation is valid for all solutions that start in Λ, it is also valid for the solution which starts in Λ and maximizes the left hand side of the inequality. In other words

$$\max_{x\in\Omega_B} \sqrt{x^T(t)\,P\,x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r + \delta \tag{5.28}$$

On the other hand

$$\max_{x\in\Omega_B} \sqrt{x^T(t)\,P\,x(t)} = \sqrt{\lambda_{\max}(P)}\,\max_{x\in\Omega_B} \|x(t)\| \tag{5.29}$$

Hence

$$\|x(t)\| \le M = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,r + \frac{\delta}{\sqrt{\lambda_{\max}(P)}}, \quad \forall\, t \ge T + t_0, \quad \forall\, x \in \Omega_{b_R} \tag{5.30}$$

Consequently, all solutions of (5.1) are UUB with the ultimate bound M. The bound is shown in Figure 5.2. Summarizing all formally proven results, we state the following theorem.

Theorem 5.1

Let $P = P^T > 0$ and $Q = Q^T > 0$ be symmetric positive definite matrices that satisfy the Lyapunov algebraic equation (5.7), and let $V(x) = x^T P\, x$. Consider the dynamics in (5.5) and suppose that $|\varepsilon(t,x)| \le \varepsilon_{\max} < \infty$, uniformly in t, and for all $x \in S_R = \{\|x\| \le R\}$, where $R > 0$ is chosen such that $S_R = \{\|x\| \le R\} \subset \Omega$. Let $S_r = \left\{\|x\| \le r = \frac{\|P\,B\|\,\varepsilon_{\max}}{\lambda_{\min}(Q)}\right\}$ and suppose that $\frac{r}{R} < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$. Then those solutions of (5.5) that start in the bounded set $\Omega_{b_R} = \{V(x) \le b_R = \lambda_{\min}(P)\,R^2\}$ are UUB, with the ultimate bound M, as it is defined in (5.30).

6. Invariance – like Theorems

For autonomous systems, the LaSalle's invariance set theorems allow asymptotic stability conclusions to be drawn even when $\dot V$ is only negative semi-definite in a domain Ω. In that case, the system trajectory approaches the largest invariant set E, which is a subset of all points $x \in \Omega$ where $\dot V(x) = 0$. However the invariant set theorems are not applicable to nonautonomous systems. In the case of the latter, it may not even be clear how to define a set E, since V may explicitly depend on both t and x. Even when $V = V(x)$ does not explicitly depend on t the nonautonomous nature of the system dynamics precludes the use of the LaSalle's invariant set theorems.

Example 6.1

The closed-loop error dynamics of an adaptive control system for 1st order plant with one unknown parameter is

$$\dot{e} = -e + \theta\, w(t)$$

$$\dot{\theta} = -e\, w(t)$$

where e represents the tracking error and $w(t)$ is a bounded function of time t. Due to the presence of $w(t)$, the system dynamics is nonautonomous. Consider the Lyapunov function candidate

$$V(e,\theta) = e^2 + \theta^2$$

Its time derivative along the system trajectories is

$$\dot{V}(e,\theta) = 2\,e\,\dot{e} + 2\,\theta\,\dot{\theta} = 2\,e\big(-e + \theta\, w(t)\big) + 2\,\theta\big(-e\, w(t)\big) = -2\,e^2 \le 0$$

This implies that V is a decreasing function of time, and therefore, both $e(t)$ and $\theta(t)$ are bounded signals of time. But due to the nonautonomous nature of the system dynamics, the LaSalle's *invariance set theorems cannot be used* to conclude the convergence of $e(t)$ to the origin.

In general, if $\dot{V}(t,x) \le -W(x) \le 0$ then we may expect that the trajectory of the system approaches the set $\{W(x) = 0\}$, as $t \to \infty$. Before we formulate main results, we state a lemma that is interesting in its own sake. The lemma is an important result about asymptotic properties of functions and their derivatives and it is known as the Barbalat's lemma. We begin with the definition of a uniform continuity.

Definition 6.1 (uniform continuity)

A function $f(t): R \to R$ is said to be *uniformly continuous* if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta = \delta(\varepsilon) > 0 \quad \forall\, |t_2 - t_1| \le \delta \Rightarrow |f(t_2) - f(t_1)| \le \varepsilon$$

Note that t_1 and t_2 play a symmetric role in the definition of the uniform continuity.

Lemma 6.1 (Barbalat)

Let $f(t): R \to R$ be differentiable and has a finite limit, as $t \to \infty$. If $\dot{f}(t)$ is uniformly continuous then $\dot{f}(t) \to 0$, as $t \to \infty$.

Lemma 6.2

If $\dot{f}(t)$ is bounded then $f(t)$ is uniformly continuous.

An immediate and a very practical corollary of Barbalat's lemma can now be stated.

Corollary 6.1

If $f(t): R \to R$ is twice differentiable, has a finite limit, and its 2nd derivative is bounded then $\dot{f}(t) \to 0$ as $t \to \infty$.

In general, the fact that derivative tends to zero does not imply that the function has a limit. Also, the converse is not true. In other words:

$$f(t) \to C \not\Leftrightarrow \dot{f}(t) \to 0$$

Example 6.2

- As $t \to \infty$, $f(t) = \sin(\ln t)$ does not have a limit, while $\dot{f}(t) = \dfrac{\cos(\ln t)}{t} \to 0$.
- As $t \to \infty$, $f(t) = e^{-t} \sin\left(e^{2t}\right) \to 0$, while $\dot{f}(t) = -e^{-t} \sin\left(e^{2t}\right) + e^{t} \cos\left(e^{2t}\right) \to \infty$.