Project Ideas: Lagrangian Coherent Structures

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A. Prove the Flux Estimate Theorem:

1. Please refer to the paper: S. C. Shadden, F. Lekien, and J. E. Marsden, *Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows*, Physica D: Nonlinear Phenomena, **212**(3-4), 271–304 (2005).
   Starting with the definition of LCS provided in Definition 2.3, provide all the needed calculations to prove the flux estimate provided in Theorem 4.4. Of course, you can reproduce the proofs as they are given in the paper, but try to provide your own commentary to the calculations that show that you understand the progression of the proof. Do not simply regurgitate the proof as it is given.
2. *Optional:* Consider the following condition that must be true for points on the LCS:

$$\nabla\sigma(\mathbf{x}_{LCS}, t) \cdot \hat{\mathbf{n}}(\mathbf{x}_{LCS}, t) = 0\,. \tag{1}$$

   By implicitly differentiating this condition with respect to time, obtain an expression for the motion of the LCS. Can you use this information to provide a much shorter proof of the flux estimate (or similar type of flux estimate)?

B. FTLE computation

1. Write your own code to numerically compute the FTLE for the following dynamical system in the plane:

$$\dot{x} = y \tag{2}$$

$$\dot{y} = -\sin(x) - 0.65y\sin(\pi t)\,. \tag{3}$$

2. Plot the FTLE using a flooded contour plot.
3. Animate many frames of the FTLE at different times to produce a movie of the FTLE.
4. Plot instantaneous streamlines of the velocity field and compare them with the LCS. Do instantaneous streamlines provide information about transport barriers in the flow?
5. *Optional:* Propose and implement an algorithm to extract the LCS as a curve defined by line segments from the FTLE scalar field. Will your algorithm work in 3 dimensions?