Hamiltonian Dynamics
CDS 140b

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CDS

Feb. 10, 2009
Outline

1. Introductory concepts;
2. Poisson brackets;
3. Integrability;
4. Perturbations of integrable systems.
References

- J. D. Meiss: *Visual Exploration of Dynamics: the Standard Map*. See http://arxiv.org/abs/0801.0883 (has links to software used to make most of the plots in this lecture)


- GniCodes: symplectic integration of 2nd order ODEs. Similar in use as Matlab’s ode suite. See http://www.unige.ch/~hairer/preprints/gnicodes.html
Hamilton’s equations

A Hamiltonian is a function $H : \mathbb{R}^{2N} \to \mathbb{R}$.

- Variational interpretation: extrema of the following action functional:

$$S(q(t), p(t), t) = \int p_i(t) \dot{q}^i(t) - H(q(t), p(t), t) dt,$$

where $(q^i, p_i)$ are coordinates on $\mathbb{R}^{2N}$.

- Equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

- In this form: defined on a $2n$-dimensional phase space (cotangent bundle).

$$i_X \omega = dH.$$
Eigenvalues of the linearization

If $\lambda$ is an eigenvalue of $D^2H$, then so are $-\lambda$ and $\lambda^*$. 

- No sources or sinks; 
- No Hopf bifurcation in the classical sense.
Poisson brackets: definition

Let $f(q, p, t)$ be a time-dependent function on phase space. Its total derivate is

$$\dot{f} \equiv \frac{df}{dt} = \frac{\partial f}{\partial q^i} \frac{dq^i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial f}{\partial t}$$

$$= \{f, H\} + \frac{\partial f}{\partial t},$$

where we have defined the (canonical) Poisson bracket of two functions $f$ and $g$ on phase space as

$$\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}.$$
Poisson brackets: properties

A Poisson bracket is an operation \( \{ \cdot, \cdot \} \) on functions satisfying the following properties:

1. \( \{ f, g \} = - \{ g, f \} \);
2. \( \{ f + g, h \} = \{ f, h \} + \{ g, h \} \);
3. \( \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \);
4. \( \{ fg, h \} = f \{ g, h \} + g \{ f, h \} \).

Property 3 is called the Jacobi identity. Properties 1, 2, and 3 make the ring of functions on \( \mathbb{R}^{2n} \) into a Lie algebra.
Rewriting Hamilton’s equations

▶ For any function $f$ on phase space, we have

$$\frac{df}{dt} = \{f, H\}.$$ 

▶ For $f = q^i$ and $f = p_i$, we recover Hamilton’s equations:

$$\dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i}.$$ 

▶ This definition makes sense for any Poisson bracket.
Not all Poisson brackets are canonical: Euler equations

Consider a rigid body with moments of inertia \((I_1, I_2, I_3)\) and angular velocity \(\Omega = (\Omega_1, \Omega_2, \Omega_3)\). Define the angular momentum vector

\[
\Pi = (\Pi_1, \Pi_2, \Pi_3) = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3).
\]

The equations of motion for the rigid body (Euler equations) are

\[
\dot{\Pi} = \Pi \times \Omega.
\]

or, written out in components,

\[
\begin{align*}
I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3, \\
I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1, \\
I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2.
\end{align*}
\]

Clearly not canonical, since odd number of equations.
Define the rigid body Poisson bracket on functions \( F(\Pi), G(\Pi) \) as

\[
\{ F, G \}_{r.b.}(\Pi) = -\Pi \cdot (\nabla F \times \nabla G).
\]

The Euler equations are equivalent to

\[
\dot{F} = \{ F, H \}_{r.b.},
\]

where the Hamiltonian is given by

\[
H = \frac{1}{2} \left( \frac{\Pi_1^2}{l_1} + \frac{\Pi_2^2}{l_2} + \frac{\Pi_3^2}{l_3} \right).
\]
Conserved quantities

**Definition**

A function $f$ is a **conserved quantity** if it Poisson commutes with the Hamiltonian:

$$\{f, H\} = 0.$$ 

**Immediate consequences:**

- If the Hamiltonian is autonomous, then it is conserved, as $\{H, H\} = 0$;
- If $f$ and $g$ are conserved, then so is $\{f, g\}$:
  $$\{\{f, g\}, H\} = \{\{g, H\}, f\} - \{\{f, H\}, g\} = 0,$$
  using the Jacobi identity. Usually this doesn’t give too much information.
Characteristic property of Hamiltonian flows

Theorem

The flow of a Hamiltonian vector field preserves the Poisson structure:

\[ \{ F, G \} \circ \Phi_t = \{ F \circ \Phi_t, G \circ \Phi_t \}. \]

Take the derivative of \( u := \{ F \circ \Phi_t, G \circ \Phi_t \} - \{ F, G \} \circ \Phi_t \):

\[ \frac{du}{dt} = \{ \{ F \circ \Phi_t, H \}, G \circ \Phi_t \} + \{ F \circ \Phi_t, \{ G \circ \Phi_t, H \} \} - \{ \{ F, G \} \circ \Phi_t, H \}. \]

- Jacobi identity: \( \frac{du}{dt} = \{ u, H \} \).
- Solution: \( u_t = u_0 \circ \Phi_t \), but \( u_0 = 0 \).
Liouville’s theorem

To each Hamiltonian $H$, associate the following vector field on phase space:

$$X_H := \left( \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \right)^T.$$ 

Liouville’s theorem

The flow of $X_H$ preserves volume in phase space.

- Consequence of the following fact:

  flow of $\dot{x} = f(x)$ is volume preserving $\iff \text{div} f(x) = 0$.

- 

  $$\text{div}X_H = \frac{\partial}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q^i} = 0.$$ 

  More is true: the flow of $X_H$ is symplectic.
Consequences of Liouville’s theorem

1. In a Hamiltonian system, there are no asymptotically stable equilibria or limit cycles in phase space.

2. Poincaré’s recurrence theorem:

Theorem

Let $\Phi : D \rightarrow D$ be a volume preserving diffeomorphism from a bounded region $D \subset \mathbb{R}^n$ to itself. Then for any neighborhood $U$ in $D$, there is a point of $U$ returning to $U$ after sufficiently many iterations of $\Phi$. 
Integrability: definitions

Definition
A Hamiltonian system with \( n \) degrees of freedom and Hamiltonian \( H \) is called integrable if there exist \( n \) functionally independent integrals \( F_i, i = 1, \ldots, n \), which are in involution: \( \{ F_i, F_j \} = 0 \).

The odds of randomly picking an integrable Hamiltonian among the class of all analytic functions are zero.

Motivation: if system is integrable, one can find a canonical trafo to action-angle coordinates

\[
(p, q) \mapsto (I, \phi) \quad \text{and} \quad H(p, q) \mapsto H'(I),
\]

i.e. \( H' \) doesn’t depend on \( \phi \). Trivially integrable eqns of motion:

\[
\dot{l} = 0 \quad \text{and} \quad \dot{\phi} = \frac{\partial H'}{\partial l} := \omega(I).
\]
Integrability: examples

▶ Simple example

\[ H = \frac{1}{2}(p_1^2 + q_1^2) + \frac{1}{2}(p_2^2 + \omega q_2^2) \]

Integrable; action-angle form: \( H' = I_1 + I_2\omega \). Trajectories fill out 2-torus if \( \omega \) is irrational.

▶ Rigid body (energy and angular momentum conserved);
▶ Kepler problem (energy, Laplace-Runge-Lenz vector, angular momentum), etc.
The Arnold-Liouville theorem

Let $M_f$ be a level set of the $F_i$:

\[ M_f := \{ x : F_i(x) = f_i, i = 1, \ldots, n \}. \]

**Theorem**

- If $M_f$ is compact and connected, then it is diffeomorphic to a smooth $n$-torus;
- $M_f$ is invariant under the flow of $H$, and the motion on $M_f$ is quasi-periodic. In angular coordinates: $\dot{\phi}_i = \omega_i(f)$.

Important to remember: integrability $\rightarrow$ motion on invariant tori.
Proof of Arnold’s theorem

Idea:

1. the $n$ conserved quantities $F_i$ generate $n$ commuting vector fields $X_{F_i}$;
2. composition of the flows of the $X_{F_i}$ defines an action of $\mathbb{R}^n$ on $M$;
3. the isotropy subgroup of each point is a lattice in $\mathbb{R}^n$. 
Perturbed Hamiltonian systems

**Question:** what can we say about systems with Hamiltonians of the form

\[ H = H_0 + \epsilon H_1, \]

where \( H_0 \) is integrable, and \( \epsilon \) is small?

**Possible answers**

- Since perturbation is small, the resulting dynamics will still be close to the original dynamics (Birkhoff averaging);
- Even a tiny perturbation destroys integrability completely, rendering the system ergodic (Fermi’s point of view).

**KAM theory:** sometimes one is true, in some cases the other → very rich picture!
Importance of near integrability

Many examples in practice

- special cases of 3-body problem;
- Motion of a charged particle in a tokamak;
- The Hénon-Heiles potential (astronomy again).

The solution

KAM theorem (after Kolmogorov-Arnold-Moser)

- One of the most important theorems in 20th century mathematical physics;
- Many deep connections with other branches of math, like number theory, analysis, etc.

Disclaimer: KAM is extremely technical! Fortunately, computer simulations give a quick and far-reaching insight (project idea).
Understanding the KAM theorem

Recall that the phase space of an integrable system is (locally) foliated by tori.

The KAM theorem:
- The tori with rational rotation numbers are destroyed under arbitrary perturbations.
- Those with “very irrational” rotation numbers persist.
Some definitions

Definition

- **Resonant torus**: one for which the rotation numbers \((\omega_1, \ldots, \omega_n)\) satisfy

  \[ k_1\omega_1 + \cdots + k_n\omega_n = 0, \]

  for some \((k_1, \ldots, k_n) \in \mathbb{Z}^n\). Otherwise, torus is non-resonant.

- **Strongly non-resonant**: there exist \(\alpha > 0\) and \(\tau > 0\) such that

  \[ |k_1\omega_1 + \cdots + k_n\omega_n| > \frac{\alpha}{|k|^\tau} \]

  for all \(k \in \mathbb{Z}^n, k \neq 0\).

Denote the set of all strongly non-resonant frequencies by \(\Delta_{\tau}^{\alpha}\).
Area-preserving mappings

There is a one-to-one correspondence between 2D Hamiltonian systems and area preserving mappings of the plane.
Integrable systems: twist mappings

For an integrable system, the Poincaré map $P$ is a twist map: $P$ rotates circles in opposite directions.
The fate of resonant tori

Let $T$ be an area preserving mapping. So $T$ can be

1. a Poincaré mapping;
2. the time-$\tau$ advance mapping of some non-autonomous Hamiltonian flow;
3. just any area preserving map.

Consider a 1D torus $\mathcal{C}$ with rational rotation number: $\omega_1/\omega_2 = r/s \in \mathbb{Q}$. Note that every point of $\mathcal{C}$ is a fixed point of $T^s$.

Poincaré-Birkhoff

Under small perturbations: the resonant torus breaks up and leaves $2ks$ fixed points of $T^s$ in its wake, which are alternatingly elliptic and hyperbolic.

Technical condition: $T$ should be a twist mapping.
The fate of resonant tori (2)

The following picture “illustrates” the proof of Poincaré-Birkhoff:

![Diagram of resonant tori]

The full proof (very intuitive!) can be found in Tabor (p. 141) or Verhulst (p. 236).
Small perturbations

- So... all resonant tori are destroyed under arbitrarily small perturbations and give rise to chaos. Does that mean that the system is ergodic?

- Answer: an emphatic no. Let's find out what happens to the non-resonant tori!
The fate of the strongly non-resonant tori

Structure of the sets $\Omega_{\alpha}^\tau$

Here $\Omega_{\alpha}^\tau := \Delta_{\alpha}^\tau \cap \Omega$, with $\Omega \subset \mathbb{R}^n$ compact.

**Theorem**

For all $\alpha$ and $\tau > n - 1$, $\Omega_{\alpha}^\tau$ is a Cantor set. The complement of $\Omega_{\alpha}^\tau$ has Lebesgue measure of the order $\alpha$.

Define

$$R_{\alpha,k}^\tau = \left\{ \omega \in \Omega : |k_1 \omega_1 + \cdots + k_n \omega_n| < \frac{\alpha}{|k|\tau} \right\}.$$  

Then $\bigcup_{0 \neq k \in \mathbb{Z}^n} R_{\alpha,k}^\tau$ is the complement of $\Omega_{\alpha}^\tau$.

- $R_{\alpha}^\tau$ is open;
- $R_{\alpha}^\tau$ is dense in $\Omega$ (since $\mathbb{Q}^n \cap \Omega \subset R_{\alpha}^\tau$).

Sufficient to have a Cantor set.
Cantor’s set

- Remove “middle thirds” from the unit interval.

- \( \dimhd C = \frac{\log 2}{\log 3} < 1 \).

- “Almost all” points lie outside \( C \).
The KAM theorem

Roughly speaking: under suitable conditions of nondegeneracy, the following holds:

▶ For perturbations of the order $\alpha^2$, all tori in $\Omega_{\alpha}^\tau$ persist;
▶ The destroyed tori fill an part of phase space with measure of order $\alpha$. 
Project ideas

▶ KAM theory
  Article by J. Meiss.
▶ Homoclinic tangles and chaos.
  Michael Tabor, *Chaos and Integrability in Nonlinear Systems*.
▶ Dynamics of point vortices and chaotic motions.
▶ In-depth study of the Lorentz attractor.
  Devaney et al., *Differential Equations, Dynamical Systems & An Introduction to Chaos*. 