CDS 140b: Homework Set 3

Due by Tuesday, February 10, 2009.

1. *Structural stability and topological equivalence.*

   Consider the systems

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= x \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} &= -y + \mu x \\ \dot{y} &= x + \mu y \end{cases}$$

   for $\mu \neq 0$.

   (a) Let $F_t, G_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the flows defined by these two systems. Show that for $\mu < 0$

$$\lim_{t\to+\infty} F_t(x) \neq 0 \quad \text{and} \quad \lim_{t\to+\infty} G_t(x) = 0$$

   for all $x \in \mathbb{R}^2$, $x \neq 0$.

   (b) Show that both systems are topologically inequivalent for $\mu < 0$. Hint: assume to the contrary that there exists a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a strictly nondecreasing function $\tau \mapsto t(\tau)$ from $\mathbb{R}$ to itself such that

$$F_{t(\tau)} = \phi^{-1} \circ G_\tau \circ \phi.$$

   Use this expression together with the limit properties derived in (a) to arrive at a contradiction.

2. *A variation on a problem from the 2008 final for CDS-140a.*

   A bead of mass m can slide along along the vertical and is attached to two walls by means of identical springs with spring constant k and natural length l. The distance between each of the walls and the vertical is given by b. There is no gravity but there is some damping with damping constant μ. The distance x between the bead and the horizontal will obey the following equation of motion:

$$\ddot{x} = -2\left(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}\right) x - \mu \dot{x},$$

   where without loss of generality we have set $k = m = 1$ and $\alpha = b/l$. Note that $\mu > 0$.

   (a) Convert the equation to a system of first-order ODEs. Derive an expression for the equilibrium points and their stability in function of α.

   (b) Sketch the bifurcation diagram, where the value of α is plotted on the horizontal axis and the x-values of the critical points on the vertical axis.

   (c) By inspecting the bifurcation diagram, determine the nature of the bifurcation that occurs when $\alpha = 1$.

3. *Bifurcations and index theory.*

   Consider the following two-dimensional version of the saddle-node bifurcation:

$$\dot{x} = a + x^2 \quad \text{and} \quad \dot{y} = -y,$$

   where $a \in \mathbb{R}$.

   (a) Find and classify all the fixed points for $a \in \mathbb{R}$. Sketch the phase portrait for $a < 0$, $a = 0$, and $a > 0$.

(b) For $a = 0$, argue that the index of the fixed point at the origin is zero.

(c) Show that the sum of the indices of the fixed points is conserved as a varies. Make sure to include the case $a = 0$.

One can hence think of the index of a fixed point as being similar to "charge": fixed points can be created or annihilated in a bifurcation but the total index remains conserved.

4. *Bifurcations in 2D.*

Consider the two-dimensional system

$$\begin{aligned}\dot{x} &= -x^4 + 5\mu x^2 - 4\mu^2 \\ \dot{y} &= -y.\end{aligned}$$

(a) Determine the critical points and the bifurcation diagram for the system. Draw the phase portraits for various values of μ.

(b) Count the indices of the fixed points and show that the sum of all the indices is conserved as μ varies.

5. *The Hopf bifurcation.*

Show that the system

$$\begin{aligned}\dot{x} &= \mu x - y - (x^2 + y^2)x + (x^2 + y^2)y \\ \dot{y} &= x + \mu y - (x^2 + y^2)x - (x^2 + y^2)y\end{aligned}$$

exhibits a Hopf bifurcation at the origin at the bifurcation value $\mu = 0$. Find an explicit expression for the limit cycle of this system and discuss its stability in function of μ. Is this Hopf bifurcation supercritical or subcritical?