CDS 140b: Homework Set 2

Due by Thursday, January 29, 2009.

1. Consider the system

$$\dot{x} = -y + x(r^4 - 3r^2 + 1)$$
$$\dot{y} = x + y(r^4 - 3r^2 + 1)$$

where $r^2 = x^2 + y^2$.

(a) Show that $\dot{r} < 0$ on the circle $r = 1$ and $\dot{r} > 0$ on $r = 2$. Use the Poincaré-Bendixson theorem to show that there exists a limit cycle in the annular region $A_1 = \{(x, y) : 1 < r < 2\}$.
Hint: for this part of the exercise, a slightly modified version of the Poincaré-Bendixson theorem has to be used. See remark 1 in the notes.

(b) Show that the origin is an unstable fixed point of the system and use this together with the fact that $\dot{r} < 0$ on $r = 1$ to conclude (again by the Poincaré-Bendixson theorem) that there exists a limit cycle in the annular region $A_2 = \{(x, y) : 0 < r < 1\}$.

(c) Convert the system to polar coordinates, find the explicit expression for both limit cycles and discuss their stability.

2. Show that the Liénard system defined by

$$g(x) = x \quad \text{and} \quad F(x) = \frac{x^3 - x}{x^2 + 1},$$

i.e. $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$, has a unique stable limit cycle.

3. Write out the details for the Poincaré-Lindstedt method applied to the Duffing oscillator as described in the notes (§6).

(a) Introduce a new time scale $\tau = \omega t$ and rewrite the ODE in this new time scale.

(b) Introduce series expansions for x and ω and derive equations for x_0 and x_1. What are the initial conditions for x_0 and x_1?

(c) Solve the equation for x_1 and use the freedom in the series expansion for ω to eliminate secular terms. Hint: use the trigonometric identity

$$\cos^3 \tau = \frac{3\cos\tau + \cos 3\tau}{4}.$$

(d) Show that the zeroth-order solution for the Duffing oscillator is given by

$$x_0(t) = a\cos(1 + \epsilon\frac{3}{8}a^2)t.$$

4. Apply the averaging method to the Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = 0$$

to find a first approximation (*i.e.* the x_0 term in the expansion) to the true solution. First, derive the averaged equations. Then, integrate these equations for a general set of initial conditions and compare the solution with the approximation obtained in class using the Poincaré-Lindstedt method.