Syllabus: Bifurcations

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1 Structural Stability

Recall that a **homeomorphism** is a continuous bijection with a continuous inverse.

Definition 1.1. *Let U be a domain in $\mathbb{R}^n$. Consider vector fields $X, Y \in \mathfrak{X}^1(U)$ with flows $F^X, F^Y : I \times U \to U$, where I is an open interval in $\mathbb{R}$ such that $0 \in I$. The dynamics of X and Y is said to be C^0-**equivalent** if there exists a homeomorphism $\phi : U \to U$ with the following property: for all $t \in I$ there exists a $t' \in I$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "F_t^X"] \arrow[d, "\phi"'] & U \arrow[d, "\phi"] \\ U \arrow[r, "F_{t'}^Y"'] & U \end{tikzcd}$$

We will only apply this definition on vector fields defined in the plane. Intuitively, the idea is that ϕ maps the phase plane of X into the phase plane of Y, possibly distorting the trajectories (in a continuous way).

Definition 1.2. *The C^1-**norm** of a vector field $X \in \mathfrak{X}^1(U)$ is defined as*

$$\|X\|_1 = \sup_{x \in U} \|X(x)\| + \sup_{x \in U} \|DX(x)\|.$$

If U is compact, then $\|X\|_1 < +\infty$.

The C^1-norm provides us with a notion of "nearness" in the space of vector fields, making it into a Banach space. Using this notion, we say that a vector field is *structurally stable* if all nearby vector fields are topologically equivalent to the original vector field.

Definition 1.3. *A vector field $X \in \mathfrak{X}^1(U)$ is said to be **structurally stable** if there exists an $\epsilon > 0$ such that the following property holds: for all $Y \in \mathfrak{X}^1(U)$ and $K \in U$ a compact set, if $\|X - Y\|_1 < \epsilon$ on K, then X and Y are topologically equivalent on K.*

This definition is different from the one used in Perko [2001] but it agrees with Holmes et al. [1996]. However, Perko [2001] uses it implicitly, see for instance section 4.1, example 1 and 2 (p. 319).

Example 1. The harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$ is structurally unstable. Indeed, it is a linear system with eigenvalues on the imaginary axis, so arbitrarily small linear perturbations will turn the origin into a source or a sink.

Example 2. Consider the following perturbation of the harmonic oscillator:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \epsilon x^2.\end{aligned}$$

The origin is still a center but the nonlinear perturbation term introduces a new, hyperbolic fixed point at $(1/\epsilon, 0)$ and a homoclinic orbit encircling the origin. This is another example showing that the harmonic oscillator is structurally unstable.

However, for this to be a "small perturbation" of the harmonic oscillator, we need to check that it is close to the harmonic oscillator *on compact subsets of* $\mathbb{R}^2$. Indeed, note that the term ϵx^2 is unbounded on $\mathbb{R}^2$, no matter how small ϵ.

Generic Properties. A property is typically said to be *generic* if "almost all" systems have that property. It remains to define "almost all": in measure theory, this can be done by saying that the complement of the generic set has measure zero. In topology and dynamical system, a slightly different definition is used. We stick with Perko [2001] and define a set to be **generic** if it contains an open and dense subset.

Peixoto's Theorem. In two dimensions, it is possible to give a full classification of structurally stable systems. This is the content of Peixoto's theorem. Roughly speaking, a dynamical system on a compact, two-dimensional manifold is structurally stable iff it has the following properties:

1. The system has a finite number of fixed points and limit cycles and all are hyperbolic;
2. There are no heteroclinic or homoclinic orbits;
3. The nonwandering set contains fixed points and limit cycles only.

If the manifold is orientable (meaning that a continuous, consistent choice of normal vector can made — unlike (say) for the Möbius strip), the set of structurally stable vector fields is generic in the set of all vector fields, equipped with the C^1-topology. See Perko [2001] for more information.

It is relatively straightforward to find structurally unstable systems that do not satisfy some of the properties above. We've already seen that linear systems with centers are structurally unstable. To show that separatrices lead to structural instability, it suffices to think of the pendulum, where a homoclinic orbit connects the unstable upper equilibrium to itself. The slightest amount of dissipation or forcing destroys this separatrix. Finally, to exhibit the importance of the wandering set, consider the linear flow on the torus[1] given by

$$\dot{x} = 1 \quad \text{and} \quad \dot{y} = a.$$

For $a \in \mathbb{Q}$, all orbits on the torus are periodic, while for $a \in \mathbb{R} - \mathbb{Q}$ the orbits fill up the torus densely (the orbits are said to be **quasiperiodic**). In both cases, the nonwandering set consists of the entire torus. However, the dynamical system is not structurally stable: since both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, an arbitrary small perturbation will cause the system to go from genuinely periodic to quasiperiodic behavior, and vice versa.

[1] Here, we've identified the torus with the unit-length square in the plane with opposite sides identified.

2 Codimension-one Bifurcations

2.1 Bifurcations at Nonhyperbolic Fixed Points

We have seen that systems with hyperbolic fixed points only are very likely to be structurally stable. Therefore, interesting phenomena will most likely occur at nonhyperbolic fixed points.

One-dimensional Systems. Let $\dot{x} = f(x, \mu)$ be a one-dimensional dynamical system ($x \in \mathbb{R}$, $\mu \in I$ where I is an open interval containing zero). We assume that the origin is a nonhyperbolic fixed point:

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) = 0.$$

In class, we discussed the following types of elementary bifurcations. See also Wiggins [1990], §3.1A.

- *The saddle-node bifurcation.* In order for this bifurcation to occur, the following should hold:

  $$\frac{\partial f}{\partial \mu}(0,0) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0.$$

  The normal form for this bifurcation is $f(x, \mu) = \mu \pm x^2$.

- *The transcritical bifurcation.* Here, we established the following conditions:

  $$\frac{\partial f}{\partial \mu}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0.$$

  The normal form is $f(x, \mu) = \mu x \pm x^2$.

- *The pitchfork bifurcation.* Here, we have

  $$\frac{\partial f}{\partial \mu}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 0, \quad \frac{\partial^3 f}{\partial x^3}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0.$$

  The normal form is $f(x, \mu) = \mu x \pm x^3$. Depending on the sign of the cubic term in the normal form, different scenarios occur: if

  $$\frac{\partial^3 f}{\partial x^3}(0,0) < 0,$$

  the bifurcation is termed **supercritical**. In the other case, the bifurcation is said to be **subcritical**.

Notice that sometimes a vector field can have a symmetry which prevents a given bifurcation from occurring. For instance, if $f(\mu, -x) = -f(\mu, x)$ for all x, then the transcritical bifurcation cannot occur. The idea of linking symmetries of the vector field with possible bifurcations is a very powerful one.

Higher-dimensional Systems. For higher-dimensional systems, we limit ourselves to discussing one possible approach using center manifolds. The full story can be found in Guckenheimer and Holmes [1990], chapter 3.

Consider a dynamical system in $\mathbb{R}^n$ of the form

$$\dot{x} = f(\mu, x), \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. Again, we assume the origin to be a nonhyperbolic fixed point; *i.e.* $f(0,0) = 0$ and the linearization $Df(0,0)$ has at least one zero eigenvalue. The case of a pair of imaginary eigenvalues will be treated later.

Consider the case of a single zero eigenvalue. Roughly speaking, we can then find a two-dimensional center manifold for the augmented system

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} f(\mu, x) \\ 0 \end{pmatrix}.$$

Note that since f has a single zero eigenvalue, the augmented system has two zero eigenvalues. The center manifold theorem then states that the dynamics can be reduced to a two-dimensional center manifold M with the following properties:

1. The tangent space to M at $(0,0)$ is spanned by the eigenvector of $Df(0,0)$ with eigenvalue zero and a vector in the μ-direction;
2. The manifold M is locally C^k and is invariant under the flow of (2.1).

Hence, one can show that studying bifurcations in higher dimensions ultimately reduces to (for fixed μ) a one-dimensional problem. In theory, one can compute the center manifold using the techniques introduced in CDS-140b. In practice, it will be "obvious" which direction corresponds to the center manifold: we will mostly just be concerned with a limited number of planar systems.

Let us illustrate this on a standard planar system:

$$\dot{x} = \mu - x^2 \quad \text{and} \quad \dot{y} = -y.$$

The augmented system has a fixed point at the origin in (x, y, μ)-space, and the center subspace is the $y = 0$ plane. It is easy to check that this plane is also invariant under the flow of the nonlinear system, which makes it into the center manifold of the augmented nonlinear system. In the $y = 0$ plane we then get a reduced dynamical system given by $\dot{x} = \mu - x^2$ and $\dot{\mu} = 0$, which is the normal form for the saddle-node bifurcation. The topological behavior of the system is therefore completely determined by the reduced system: along the (unstable) y-direction we merely get contraction towards the center manifold.

Sotomayor introduced necessary conditions for elementary bifurcations to occur in higher dimensions without recourse to center manifold theory; see Perko [2001] for a statement of his theorem.

2.2 Unfoldings of Bifurcations

We can ask what happens to a given bifurcation when additional perturbations are added to the system. For example, the flexible beam with a given load is the prime example of a pitchfork bifurcation. However, when the load is applied off-center, or when the beam has an initial non-zero curvature, the bifurcation picture is altered.

This leads us naturally to the concept of an **unfolding** of a structurally unstable vector field $f_0(x)$. This is a family of vector fields $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, $f = f(x, \lambda)$ such that $f(x, 0) = f_0(x)$. Roughly speaking, an unfolding is said to be **universal** if every other unfolding is topologically equivalent to it. Here, one must take care to ensure that the topological equivalence does not mix the dynamical variables x with the parameters λ. This leads us to the concept of **contact transformations**.

Let us illustrate this on the pitchfork bifurcation. For $\mu = 0$ we have the structurally unstable vector field $\dot{x} = -x^3$. We have seen that the structure of the pitchfork is determined by the derivatives of the vector field up to third order. Let us therefore propose the following unfolding:

$$\dot{x} = \mu_1 + \mu_2 x - x^3.$$

The x^2-term has been eliminated by shifting the origin. One can show that this unfolding is indeed universal.

2.3 The Hopf Bifurcation

We now consider the case of a dynamical system $\dot{x} = f(x, \mu)$ with a fixed point at the origin for $\mu = 0$ and where a pair of (conjugate) eigenvalues crosses the imaginary axis when μ is varied (and all other eigenvalues have nonzero real part). Because of the implicit function theorem and the fact that the Jacobian $Df(0, 0)$ is non-singular, the fixed point at the origin will persist for varying μ. However, as μ crosses zero, the dimensions of the stable, unstable, and center manifolds will change, leading to a qualitatively different phase portrait. In this way, we are led to the celebrated **Hopf bifurcation**.[2]

Example 3. Example 1 in Perko [2001] is the generic example of a system with a Hopf bifurcation. This example is of the form

$$\begin{aligned}\dot{x} &= -y + x(\mu - x^2 - y^2)\\ \dot{y} &= x + y(\mu - x^2 - y^2).\end{aligned}$$

The origin is a stable fixed point for $\mu < 0$, a center for $\mu = 0$ and an unstable fixed point for $\mu > 0$. Writing the equations in polar form gives us a better view of the qualitative features of the phase portrait: $\dot{r} = r(\mu - r^2)$, $\dot{\theta} = 1$, and for $\mu > 0$ a stable limit cycle exists with radius $\sqrt{\mu}$:

$$\gamma : t \mapsto \sqrt{\mu}(\cos t, \sin t).$$

Characterization. The crucial theorem for the existence of Hopf bifurcations is theorem 2 in Perko [2001]. Roughly speaking, it states that if the system $\dot{x} = f(x, \mu)$ has a single pair of pure imaginary eigenvalues for $\mu = \mu_0$ (and no other eigenvalues with vanishing real part), and if these eigenvalues move across the imaginary axis with nonzero velocity, *i.e.*

$$\frac{d}{d\mu}\mathrm{Re}\lambda(\mu)\Big|_{\mu=\mu_0} \neq 0,$$

then there exists a unique two-dimensional center manifold and a change of coordinates such that the system takes on the following normal form:

$$\begin{aligned}\dot{x} &= \mu x - y + ax(x^2 + y^2) - by(x^2 + y^2) + O(\|x\|^4)\\ \dot{y} &= x + \mu y + bx(x^2 + y^2) + ay(x^2 + y^2) + O(\|x\|^4).\end{aligned}$$

[2] After E. Hopf, not the topologist H. Hopf. The name "Poincaré-Andronov-Hopf bifurcation" is also used.

See Perko [2001] for a precise formulation.

If the critical point at the origin gives rise to a stable limit cycle, the Hopf bifurcation is said to be **supercritical**. If the limit cycle is unstable, we have a **subcritical** bifurcation. Note that the subcritical bifurcation is characterized by a stable fixed point turning into an unstable focus: this can be dramatic in practice.

If the system is in normal form as shown above, the Hopf bifurcation is subcritical if $a > 0$ and supercritical in the other case. However, for most examples that we will consider, the nature of the Hopf bifurcation can be derived more directly.

Properties. Strogatz [1994] lists the following generic properties of the supercritical Hopf bifurcation:

- The size of the limit cycle grows continuously from zero and is proportional to $\sqrt{\mu}$ for small μ.
- The frequency of the limit cycle is approximately given by the imaginary part of λ for $\mu = 0$. For μ close to zero, this is correct up to terms of the order of μ.

These properties are a useful heuristic to rule out **degenerate Hopf bifurcations**. If the limit cycle appears as soon as μ crosses the critical value, and has the properties listed above, the bifurcation is probably not degenerate. By contrast, degenerate bifurcations typically have a *family* of periodic orbits at the bifurcation, as is the case with the damped pendulum $\ddot{x} + \mu\dot{x} + \sin x = 0$.

The van der Pol oscillator. Recall the van der Pol equation given by $\dot{x} = \ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0$. By the averaging method (or alternatively, using the Poincaré-Lindstedt method), we established that this system has a unique limit cycle which is, for ϵ small, approximately circular with radius 2. In particular, the radius is independent of ϵ and the limit cycle appears as soon as $\epsilon \neq 0$.

To make this system fit our idea of a Hopf bifurcation, we introduce the following rescaled variable: $u = \sqrt{\epsilon}x$. In terms of u, the van der Pol equation becomes $\ddot{u} + u + u^2\dot{u} = \epsilon\dot{u}$, and the radius of the limit cycle is now $2\sqrt{\epsilon}$, as expected.

References

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