System Identification (ID): CDS 270 – I

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Part I: Lyapunov Stability Theory Overview

Lecture 1

1. Dynamical Systems

Read [Khalil]): Chapters 3 – 4.

We consider dynamical systems that are modeled by a *finite* number of coupled 1st order ordinary differential equations (ODE-s):

$$\dot{x} = f(t, x, u) \tag{1.1}$$

In (1.1), t denotes time and f is an n – dimensional vector field. We call (1.1) the *state equation*, refer to $x \in R^n$ as the *system state*, and $u \in R^m$ as the *control input*, (external signal). The number of the state vector components n is called the *order* of the system. Sometimes, another equation

$$y = h(t, x, u) \tag{1.2}$$

is also given, where $y \in R^p$ denotes the *system output*. Equations (1.1) and (1.2) together form the *system state space model*.

A solution $x(t)$ of (1.1) (if one exists) corresponds to a curve in state space, as t varies from and initial time to infinity. This curve is often referred to as a *state trajectory* or a *system trajectory*.

A special case of (1.1) – (1.2) is linear (*affine in the control input*) system

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, x)u \\ y &= h(t, x) \end{aligned} \tag{1.3}$$

Letting $x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T$, a special class of nonlinear continuous-time dynamics is given by systems in <u>*Brunovsky canonical form*</u>.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\dots\dots\dots \\ \dot{x}_n &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1.4}$$

For <u>*linear time-variant (LTV) systems*</u> the state space model (1.1) – (1.2) is:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned} \tag{1.5}$$

Finally, the class of <u>*linear time-invariant (LTI) systems*</u> is written in the familiar form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.6}$$

If (1.1) does not contain an input signal u

$$\dot{x} = f(t, x) \tag{1.7}$$

then the resulting dynamics is called <u>*unforced*</u>. If in addition the function f does not depend explicitly on t, that is

$$\dot{x} = f(x) \tag{1.8}$$

then the system dynamics is called <u>*autonomous*</u> or time-invariant. Systems that do depend on time (explicitly) are called <u>*non-autonomous*</u> or time-variant.

A point $x = x^*$ in the state space is an <u>*equilibrium*</u> point of (1.8) if

$$f(x^*) = 0 \tag{1.9}$$

In other words, whenever the state of the system starts at x^*, it will remain at x^* for all future times.

The linear system $\dot{x} = Ax$ has an <u>*isolated equilibrium*</u> point at $x = 0$ if $\det A \neq 0$, that is if A has no zero eigenvalues. Otherwise, the system has a continuum of equilibrium points. These are the only possible equilibrium patterns that a linear system may have. On the other hand, a nonlinear system (1.8) can have <u>*multiple*</u> isolated equilibrium points.

2. Existence and Uniqueness

For the unforced system (1.7) to be a useful mathematical model of a physical system, it must be able to predict future states of the system given its current state x_0 at t_0. In other words, the <u>*Initial Value Problem*</u> (IVP)

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

must have a <u>*unique*</u> solution.

Example 2.1
The IVP

$$\begin{aligned} \dot{x} &= \begin{cases} -1 - x^2, & x \geq 0 \\ 1 + x^2, & x < 0 \end{cases} \\ x(0) &= 0 \end{aligned}$$

has no solutions at all (show!) on the interval $0 \leq t \leq 1$.

Example 2.2
The IVP

$$\begin{aligned} \dot{x} &= x^{\frac{2}{3}} \\ x(0) &= 0 \end{aligned}$$

has <u>*infinitely many*</u> solutions, each of which is defined on R:

$$x(t) = \begin{cases} \frac{1}{27}(t-a)^3, & t < a \\ 0, & a \leq t \leq b \\ \frac{1}{27}(t-b)^3, & t > b \end{cases}$$

where $a<0$ and $b>0$ are arbitrary constants. If $a=b=0$ then there are 2 solutions: $x(t)=\frac{t^3}{27}$ and $x(t)\equiv 0$.

The <u>*existence*</u> and <u>*uniqueness*</u> of IVP can be ensured by imposing appropriate constraints on the right hand side function $f(t,x)$ in (2.1).

We start by stating a <u>*sufficient condition*</u> for the IVP problem to admit a solution which may not be necessarily unique.

Theorem 2.1 (Cauchy / Peano Existence Theorem)
If $f(t,x)$ is continuous in a closed region

$$B=\left\{(t,x):\left|t-t_0\right|\le T,\ \left\|x-x_0\right\|\le R\right\}\subseteq R\times R^n \tag{2.2}$$

where T and R are strictly positive constants, then there exists $t_0<t_1\le T$ such that the IVP has at least one continuous in time solution $x(t)$.

In other words, continuity of $f(t,x)$ in its arguments ensures that there is at least one solution of the IVP in (2.1).

The above theorem does not guarantee the uniqueness of the solution. The key constraint that yields uniqueness is the <u>*Lipschitz condition*</u>, whereby $f(t,x)$ satisfies the inequality

$$\left\|f(t,x)-f(t,y)\right\|\le L\left\|x-y\right\| \tag{2.3}$$

for all (t,x) and (t,y) in some neighborhood of (t_0,x_0). Note that in (2.3), $\|\bullet\|$ denotes any *p*-norm.

The next theorem gives a <u>*sufficient condition*</u> for the <u>*unique existence*</u> of a solution.

Theorem 2.2 (Local Existence and Uniqueness)
Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition (2.3)

$$\forall\, x,y\in B=\left\{x\in R^n:\left\|x-x_0\right\|\le r\right\},\ \forall\, t\in\left[t_0,t_1\right] \tag{2.4}$$

Then, there exists some $\delta>0$ such that the state equation $\dot{x}=f(t,x)$ with $x(t_0)=x_0$ has a unique solution over $\left[t_0,t_0+\delta\right]$.

The key assumption in the above theorem is the Lipschitz condition (2.3) which is assumed to be valid *locally*, that is in a neighborhood of (t_0, x_0) on the compact domain B in (2.4).

One may try to extend the interval of existence and uniqueness over a given time interval $[t_0, t_0+\delta]$ by taking $t_0 \triangleq t_0+\delta$ as a new initial time and $x_0 \triangleq x(t_0+\delta)$ as a new initial state. If the conditions of the theorem are satisfied at $(t_0+\delta,\ x(t_0+\delta))$ then there exist $\delta_2 > 0$ such that the IVP has a unique solution over $[t_0+\delta, t_0+\delta+\delta_2]$ that passes through the point $(t_0+\delta,\ x(t_0+\delta))$. We piece together the solutions to establish the existence of a unique solution over the interval $[t_0, t_0+\delta+\delta_2]$. This idea can be repeated to keep extending the solution. However, in general the solution cannot be extended indefinitely. In that case, there will be a maximum interval $[t_0, T)$, where the unique solution exists.

Example 2.3
The IVP

$$\dot{x} = x^2$$
$$x(0) = 1$$

has a solution

$$x(t) = -\frac{1}{t-1}$$

which is defined only for $t < 1$ and can not be extended to R. Note that the function $f(x) = x^2$ is locally Lipschitz for all $x \in R$, and as $t \to 1$ the solution has a finite escape time, that is it leaves any compact set within a finite time. The phrase "finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time.

Assuming that $f(x)$ is *globally Lipschitz*, the next theorem establishes the existence of a unique solution over any arbitrarily large interval.

Theorem 2.3 (Global existence and Uniqueness)
Suppose that f is piecewise continuous in t and *globally* Lipschitz in x, that is the function satisfies Lipschitz condition (2.3)

$$\forall\, x, y \in R^n,\ \forall\, t \in [t_0, t_1] \tag{2.5}$$

Then the IVP (2.1) has a unique solution over $[t_0, t_1]$, where the final time t_1 may be arbitrarily large.

Sufficient conditions in the above theorem are overly conservative.

Example 2.4
The IVP

$$\dot{x} = -x^3$$
$$x(0) = x_0$$

has a unique solution

$$x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}}$$

for any initial condition x_0 and for all $t \geq 0$.

Basically, if it is known that IVP has a solution that evolves on a compact domain then the solution can be extended indefinitely.

Theorem 2.4 (Global existence and Uniqueness on a compact domain)
Let $f(t, x)$ be piecewise continuous in t, locally Lipschitz in x for all $t \geq 0$ and for all x in a domain $D \subset R^n$. Let $W \subset D$ be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of the corresponding IVP lies entirely in W. Then there is a unique solution that is defined for all $t \geq 0$.

Remark: There are extensions that deal with existence and uniqueness of IVP-s whose system dynamics is discontinuous in x, (i.e., not Lipschitz).

Example 2.5 (sliding mode)
The IVP

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\operatorname{sgn}(x_1 + x_2) \end{cases}$$
$$x(0) = x_0$$

has a unique solution for any initial condition vector x_0 and for all $t \geq 0$. The solution reaches manifold $x_1 + x_2 = 0$ in finite time and "slides" down the manifold towards the origin.

3. Lyapunov Stability

Stability of equilibrium points is usually characterized in the sense of Lyapunov:

Alexander Michailovich Lyapunov, 1857-1918

- Russian mathematician and engineer who laid out the foundation of the Stability Theory
- Results published in 1892, Russia
- Translated into French, 1907
- Reprinted by Princeton University, 1947
- American Control Engineering Community Interest, 1960's

Lyapunov stability theorems give *sufficient conditions* for stability, asymptotic stability, and so on. Statements that establish necessity of these conditions are called the converse theorems.

For example, it is known that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin.

We will be mostly concern with the 2nd theorem of Lyapunov. We will use it to: a) derive stable adaptive laws for uncertain system, and b) show boundedness of the system closed-loop solutions even when the system has no equilibrium points.

Without a loss of generality, we'll study *stability of the origin* for the autonomous system:

$$\dot{x} = f(x) \tag{3.1}$$

where $f(x)$ is locally Lipschitz in x and $f(0) = 0$.

Definition 3.1 (local stability)
The equilibrium point $x = 0$ of (3.1) is

- *stable* if
$$\forall R > 0, \quad \exists\, r(R) > 0, \quad \left\{ \|x(0)\| < r \right\} \Rightarrow \left\{ \forall t \geq 0, \|x(t)\| < R \right\} \tag{3.2}$$
- *unstable* if it is not stable (write formal definition similar to (3.2))
- *asymptotically stable* if it is stable and $r = r(R)$ can be chosen such that
$$\|x(0)\| < r \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0 \tag{3.3}$$
- *marginally stable* if it is stable but not asymptotically stable, (write formal definition)
- *exponentially stable* if it is stable and
$$\exists\, r, \alpha, \lambda > 0, \quad \forall \left\{ \|x(0)\| < r \wedge t > 0 \right\}: \quad \|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t}, \tag{3.4}$$

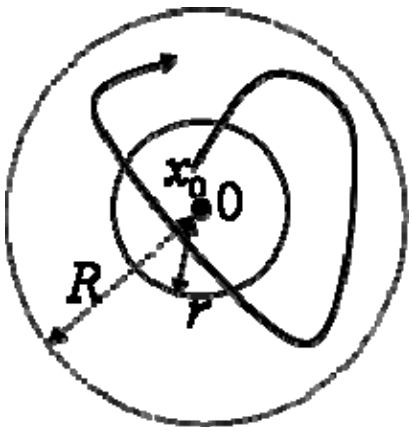

Figure 3.1: Stable System

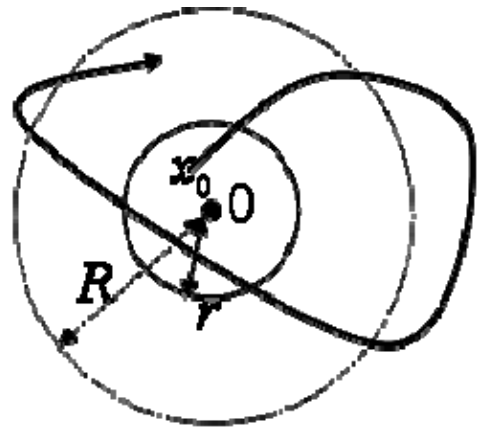

Figure 3.2: Unstable System

Basically, an equilibrium point is <u>*stable*</u> if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is <u>*asymptotically stable*</u> if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Remark: Stabilizable systems are not necessarily stable.

Note that by definition, exponential stability implies asymptotic stability, which in turn implies stability.

By definition, stability in the sense of Lyapunov defines <u>*local*</u> behavior of the system trajectories near the equilibrium. In order to analyze how the system behaves some distance away from the equilibrium, global concepts of stability are required.

Definition 3.2 (global stability)
If asymptotic (exponential) stability holds for any initial states, the equilibrium point is said to be <u>*globally*</u> asymptotically (exponentially) stable.

Next, two main theorems of Lyapunov are presented.

Theorem 3.1 (Lyapunov indirect method)
Let $x = 0$ be an equilibrium point for the nonlinear system (3.1), where $f : D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} \tag{3.5}$$

Then:

- the origin is asymptotically stable if $\operatorname{Re}\lambda_i < 0$ for all eigenvalues of A
- the origin is unstable if $\operatorname{Re}\lambda_i > 0$ for at least one of the eigenvalues of A

- if at least one of the eigenvalues is on the $j\omega$ axis, (i.e., the linearized system is marginally stable), then nothing can be said about the original nonlinear system behavior

Before stating the 2nd theorem of Lyapunov we need to introduce the concept of positive definite functions.

Definition 3.3
Let $D \subset R^n$ be a neighborhood of the origin. A function $V(x): D \to R$ is said to be:

- <u>*locally positive definite*</u>, if: $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
- <u>*locally positive semidefinite*</u>, if: $V(0) = 0$ and $V(x) \geq 0, \quad \forall x \in D - \{0\}$
- <u>*locally negative definite*</u> (<u>*semidefinite*</u>), if $-V(x)$ is locally positive definite (semidefinite)

If in the above definition $D = R^n$ then the function is <u>*globally*</u> positive (negative) definite (semidefinite).

Theorem 3.2 (Lyapunov direct method)
Let $x = 0$ be an equilibrium point for (3.1) and let $D \subset R^n$ be a domain containing the origin. If there is a <u>*continuously differentiable positive definite*</u> function $V(x): D \to R$, whose <u>*time derivative along the system trajectories*</u> is negative semidefinite in D

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \tag{3.6}$$

then the equilibrium is <u>*stable*</u>. Moreover, if $\dot{V}(x) < 0$ in $D - \{0\}$, then the equilibrium is <u>*asymptotically stable*</u>.

Definition 3.4
A continuously differentiable positive definite function $V(x)$ satisfying (3.6) is called a <u>*Lyapunov function*</u>.

Example 3.1

Consider the 1st order ODE,

$$\dot{x} = -c(x)$$

where $c(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$\{c(0) = 0\} \wedge \{c(x)x > 0, \forall x \neq 0 : x \in (-a \quad a)\}$$

One can show that both $V(x) = \int_0^x c(y)\,dy$ and $V(x) = x^2$ are the Lyapunov functions and consequently the origin is an asymptotically stable equilibrium (locally) of the system.

When the origin $x = 0$ is an asymptotically stable equilibrium of the system, we are often interested in determining its <u>*region of attraction*</u>, (also called region of asymptotic stability, domain of attraction, or basin). We want to be able to answer the question: Under what condition will the region of attraction be the whole space R^n?

Definition 3.5
If the region of attraction of an asymptotically stable equilibrium point at the origin is the whole space R^n, the equilibrium is said to be <u>*globally asymptotically stable*</u>.

Definition 3.6
A function $V : R^n \to R$ such that $\lim_{\|x\| \to \infty} V(x) = \infty$ is called <u>*radially unbounded*</u>.

Theorem 3.3 (due to Barbashin and Krasovskii)
Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a radially unbounded Lyapunov function of the system. Then the equilibrium is globally asymptotically stable.

4. LaSalle's Invariance Principle

We begin with a motivating example.

Example 4.1 (nonlinear pendulum dynamics with friction)

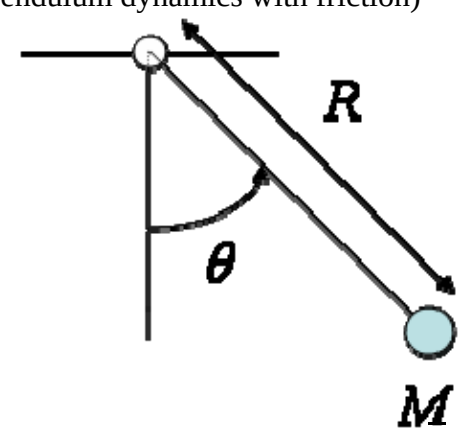

Figure 4.1: Pendulum

Dynamics of a pendulum with friction can be written as:

$$M\, R^2\, \ddot{\theta} + k\, \dot{\theta} + M\, g\, R \sin(\theta) = 0 \tag{4.1}$$

or, equivalently in state space form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b\, x_2 \end{aligned} \tag{4.2}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $a = \dfrac{g}{R}$, and $b = \dfrac{k}{M\,R^2}$. We study stability of the origin $x_e = 0$. Note that the latter is equivalent to studying stability of all the equilibrium points in the form: $x_e = \begin{pmatrix} 2\pi l & 0 \end{pmatrix}^T$, $l = 0, \pm 1, \pm 2, \ldots$ Consider the total energy of the pendulum as a Lyapunov function candidate.

$$V(x) = \underbrace{\int_0^{x_1} a \sin y \, dy}_{\text{Potential}} + \underbrace{\frac{x_2^2}{2}}_{\text{Kinetic}} = a\left(1 - \cos x_1\right) + \frac{x_2^2}{2} \tag{4.3}$$

It is clear that $V(x)$ is a positive definite function, (locally, around the origin). Its time derivative along the system trajectories is:

$$\dot{V}(x) = a \sin x_1 \, \dot{x}_1 + x_2 \, \dot{x}_2 = -b\, x_2^2 \le 0 \tag{4.4}$$

The time derivative is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1. Therefore, we can conclude that the origin is a stable equilibrium.

However, using the phase portrait of the pendulum equation (or just common sense), we expect the origin to be an asymptotically stable equilibrium. Consequently, the Lyapunov energy function argument fails to show this fact.

On the other hand, we notice that for the system to maintain $\dot{V}(x) = 0$ condition, the trajectory must be confined to the line $x_2 = 0$. Using the system dynamics (4.2) yields:

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \overset{\text{Eq. (4.2)}}{\Rightarrow} \sin x_1 \equiv 0 \Rightarrow x_1 \equiv 0$$

Hence on the segment $-\pi < x_1 < \pi$ of the line $x_2 = 0$ the system can <u>*maintain*</u> the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease to toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The forgoing argument shows that if in a domain about the origin we can find a Lyapunov function whose derivative along the system trajectories is negative semidefinite, and we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$, except at the origin, then the origin is asymptotically stable. This argument follows from the <u>*LaSalle's Invariance Principle*</u>.

Definition 4.1

A set $M \subset R^n$ is said to be

- an <u>*invariant set*</u> with respect to (3.1) if: $x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in R$
- a <u>*positively invariant set*</u> with respect to if: $x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \ge 0$

Theorem 4.1 (due to LaSalle)

Let $\Omega \subset D \subset R^n$ be a <u>*compact positively invariant set*</u> with respect to the system dynamics (3.1). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}\left(x(t)\right) \le 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $\dot{V}(x) = 0$. Let $M \subset E$ be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$, that is

$$\lim_{t\to\infty}\left(\underbrace{\inf_{z\in M}\left\|x(t)-z\right\|}_{\operatorname{dist}\left(x(t),M\right)}\right)=0$$

Notice that the inclusion of the sets in the LaSalle's theorem is:

$$M \subset E \subset \Omega \subset D \subset R^n$$

In fact, the formal proof of the theorem reveals that all trajectories $x(t)$ are bounded and approach a <u>*positive limit set*</u> $L^+ \subset M$ as $t \to \infty$. The latter may contain asymptotically stable equilibriums and stable limit cycles.