CDS 140a Midterm Examination Policy

1. The exam is due by 5pm Tuesday, November 6, 2007 in Steele 3.
2. You shall abide by the Caltech Honor Code[1] which states

   "No member of the Caltech community shall take unfair advantage of any other member of the Caltech community."

3. If you need more space than the room given, please use blank white paper, and attach to the midterm before handing it in. Make sure to clearly state which problem the extra work is associated to.
4. You shall have no other notes/textbooks other than the class notes before you, i.e., this is a closed-textbook exam. Photocopy course notes handed out in the first two weeks of the class and class homework assignments are allowed. You are not allowed to use the Internet during the exam.
5. You may use programs such as MATLAB for numerical calculations but make sure you understand what you are doing and show the relevant steps.
6. You shall not collaborate on this exam.
7. Once you have opened the exam you have 15 minutes to read the instructions and questions. You then have three hours to work the problems and you are expected to honor the three hour and 15 minute time limit. *Any work done after the time limit but* before *four hour and 15 minute since opening will be given partial credit. Any such work should be clearly labelled as such by a demarcation.* The exam must be taken in a single sitting including and a five minute break if required.
8. Violating any of the above policy amounts to taking unfair advantage of those who abide by it and hence violates the Caltech Honor Code. Conscious failure to report suspected violation is considered a violation itself. If you suspect someone of an Honor System violation, report your suspicions to the BOC Chair (undergraduates) or the GRB Chair (graduate students). It is contrary to Institute policy for instructors to deal with suspected infractions unilaterally.
9. If you have any concerns or questions regarding the exam policy, **it is your responsibility to contact the instructors and get it clarified before opening the exam.**

Good luck ☺

[1] http://www.its.caltech.edu/~grb/HonorSystem

Notation

1. The notation $(u_1, \ldots, u_n)$ denotes a n-dimensional vector with components u_i for $i = 1, \ldots, n$.

2. For a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ denotes the gradient and hessian respectively of f. If the coordinates of $\mathbb{R}^n$ are $x = (x_1, \ldots, x_n)$, then

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \tag{1}$$

Problem 1 [10 points]. Consider the following system in $\mathbb{R}$.

$$\dot{x} = -x^4 \sin(\frac{\pi}{x}) \tag{2}$$

For this system,

- Find the equilibrium points, the corresponding eigenvalues and sketch the phase potrait.
- Show that for any solution $x(t)$ to (2), $\lim_{t\to\infty} x(t) \in [-1, 1]$.
- Write down the α and ω limit sets.
- Is the origin attracting or repelling?

Problem 2 [20 points total]. Consider the following equations of motion which models a rigid body in free space with angular velocity $\omega = (\omega_1, \omega_2, \omega_3)$ in its body frame.

$$\begin{aligned} \dot{\omega}_1 &= \frac{(J_2 - J_3)}{J_1}\omega_2\omega_3 \\ \dot{\omega}_2 &= \frac{(J_3 - J_1)}{J_2}\omega_3\omega_1 \\ \dot{\omega}_3 &= \frac{(J_1 - J_2)}{J_3}\omega_1\omega_2 \end{aligned} \tag{3}$$

Here, J_i is the moment of inertial of the rigid body about its i-th axis for $i = 1, 2, 3$. For this problem, assume $J_1 > J_2 > J_3 > 0$. The axis corresponding to J_1, J_2, J_3 are called the short, middle and long axis respectively.

- **[5 points]** Calculate the linearized equations for the equilibrium points $(0, 0, 0)$ $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. These points correspond to a rigid body which is stationary, rotating about its short, middle and long axis with angular velocity 1 rad/sec respectively.

- **[5 points]** What can you conclude about the stability of the above equilibrium points using linearization?

- **[10 points]** Can you conclude stability/instability of $(0, 0, 0)$ and $(1, 0, 0)$ using Lyapunov techniques?

  [Hint: The scalar quantities $K = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)$ and $M = \frac{1}{2}(J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2)$ are first integrals. Use these quantities or an appropriate combination of them to construct the required Lyapunov functions. For example, when studying stability of $(1, 0, 0)$, construct a function $f : \mathbb{R} \to \mathbb{R}$ such that the function $V = K + f(M)$ has the property $V(1, 0, 0) = 0$, $\frac{\partial V}{\partial \omega}|_{(1,0,0)} = 0$ and $\frac{\partial^2 V}{\partial \omega^2}|_{(1,0,0)} > 0$.]

Problem 3 [30 points total].

- **[10 points]** Consider the following dynamical system in $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= y - y^3 \\ \dot{y} &= -x + x^3 \end{aligned} \tag{4}$$

  (a) Show that the system is Hamiltonian and write down the expression for the Hamiltonian $H(x, y)$.

  (b) Find the equilibrium points and the corresponding linearization. Can you conclude anything about the stability of these equilibrium points?

  (c) Show that the level curves $H(x, y)$ =constant are solution curves.

  (d) Analyze the level curves of the Hamiltonian and sketch the complete phase potrait.

- **[20 points]** Now, consider the dynamical system in $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= -y + xy \\ \dot{y} &= x + \frac{x^2 - y^2}{2} \end{aligned} \tag{5}$$

  Answer questions (a)-(d) for this system. [Hint: For part (d), consider the lines $x = 1$, $x = \sqrt{3}y - 2$ and $x = -\sqrt{3}y - 2$.]

Problem 4 [10 points]. Consider the system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned}$$

Prove that this system does or does not have a periodic orbit [Hint: look at the phase portrait to gain intuition].

Problem 5 [30 points]. Consider the linear system

$$\dot{x} = Ax$$

where A is stable ($\mathrm{Re}(\lambda_i(A)) < 0$ for all eigenvalues $\lambda_i(A)$ of A). Thus, for $Q = Q^T > 0$, there exists a $P = P^T > 0$ such that

$$PA + A^T P = -Q.$$

Let

$$\mu(Q) = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

where $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q (which is well-defined since all of the eigenvalues of Q are strictly positive and real because Q is positive definite) and $\lambda_{\max}(P)$ is the maximum eigenvalue of P.

- **[10 points]** Show that $\mu(Q)$ has the following properties:
  - $\mu(Q) = \mu(kQ)$ for all real constants $k > 0$.
  - $\mu(I) \geq \mu(Q)$ for all $Q = Q^T > 0$ such that $\lambda_{\min}(Q) = 1$.
  - $\mu(I) \geq \mu(Q)$ for all $Q = Q^T > 0$ [Hint: use the previous two facts]. Recall: For a symmetric positive definite matrix $M = M^T > 0$,

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2.$$

- **[10 points]** Since A is stable, for any solution $x(t)$ to $\dot{x} = Ax$, there exists constants $m, \alpha > 0$ such that

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)}\|x(t_0)\|.$$

  The constant α is called *an estimate of the rate of convergence*. Show that $\mu(Q)$ is an estimate of the rate of convergence. For which Q does one obtain the largest rate of convergence?

- **[10 points]** Consider the perturbed linear system:

$$\dot{x} = Ax + g(x), \tag{6}$$

  where $g(x)$ is a function satisfying $\|g(x)\| \leq \gamma\|x\|$ for some $\gamma > 0$. Show that if

$$\gamma < \mu(Q)$$

  the system (6) is exponentially stable. For which Q does one obtain the largest upper bound on γ?