CDS 140a Final Examination Policy

1. The exam is due by 5pm Wednesday, December 12, 2007 in Steele 15.
2. You shall abide by the Caltech Honor Code[1] which states

   "No member of the Caltech community shall take unfair advantage of any other member of the Caltech community."

3. If you need more space than the room given, please use blank white paper, and attach to the midterm before handing it in. Make sure to clearly state which problem the extra work is associated to.
4. You shall have no other notes/textbooks other than the class notes before you, i.e., this is a closed-textbook exam. Photocopy course notes handed out in the class and class homework assignments are allowed. You are not allowed to use the Internet during the exam.
5. You may use programs such as MATLAB for numerical calculations but make sure you understand what you are doing and show the relevant steps.
6. You shall not collaborate on this exam.
7. Once you have opened the exam you have 15 minutes to read the instructions and questions. You then have four hours to work the problems and you are expected to honor the four hour and 15 minute time limit.
8. Violating any of the above policy amounts to taking unfair advantage of those who abide by it and hence violates the Caltech Honor Code. Conscious failure to report suspected violation is considered a violation itself. If you suspect someone of an Honor System violation, report your suspicions to the instructors, the BOC Chair (undergraduates) or the GRB Chair (graduate students). It is contrary to Institute policy for instructors to deal with suspected infractions unilaterally.
9. If you have any concerns or questions regarding the exam policy, **it is your responsibility to contact the instructors and get it clarified before opening the exam.**

Good luck ⌣̈

[1] `http://www.its.caltech.edu/~grb/HonorSystem`

Problem 1 [10 points]. Which one of the following is a resonance relation for a two dimensional system with eigenvalues λ_1, λ_2? Calculate the order of the resonance if applicable.

- $\lambda_1 = 2\lambda_2$ **2.5 points**
- $4\lambda_1 = 5\lambda_2$ **2.5 points**
- $\lambda_1 + \lambda_2 = 0$ **2.5 points**
- $4\lambda_1 = 20\lambda_2$ **2.5 points**

Problem 2 [10 points]. Consider the following systems where $x, y \in R^1$.

$$\dot{x} = 3x \tag{1}$$

and

$$\dot{y} = -y \tag{2}$$

- Are the systems C^1-conjugate? Explain **3 points**
- Are the systems C^0-conjugate? Explain **3 points**
- Suppose we relax the definition of conjugacy to the following. Define two systems in R^1 with flows $\phi_t(x)$ and $\varphi_t(y)$ R-conjugate if there exists a homeomorphism $h : R^1 \to R^1$ such that $h \circ \phi_t = \varphi_{-t} \circ h$. Show that the systems given by the above two equations are R-conjugate. **4 points**

This example demonstrates that R-conjugacy cannot distinguish between a stable and an unstable equilibrium point.

Problem 3 [20 points]. Consider the following system consisting of two planar inverted pendulum on a single cart. The cart is restricted to move in a straight track. Assume the mass of cart to be M and those of the pendula to be m_1, m_2 and the respective lengths to be l_1, l_2.

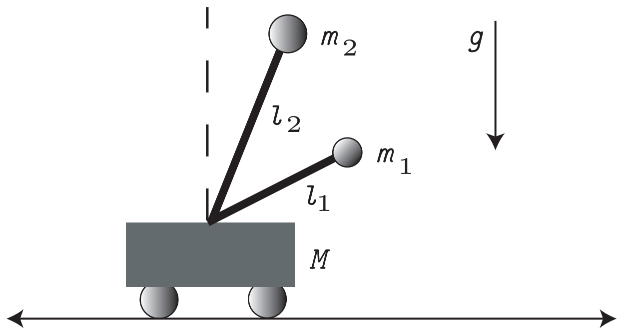

Figure 1: Two planar inverted pendula on a cart system

For this system,

- What is the configuration space Q? **2 points**
- Assuming suitable coordinate variables for Q, calculate the total kinetic energy of the system **5 points**
- Calculate the total potential energy of the system **3 points**
- Calculate the Lagrangian and the Euler-Lagrange equations of motion for the system **5 points**
- Are there any cyclic variables in the system? If so, calculate the corresponding conserved quantities. **5 points**

Problem 4 [10 points]. Consider the following system with nonzero parameters a, b. Show that the system undergoes a Poincaré-Andronov-Hopf bifurcation at $b = 1 + a^2$. Assume a to be a fixed parameter and vary b.

$$\begin{aligned} \dot{x}_1 &= a - (b+1)x_1 + x_1^2 x_2 \\ \dot{x}_1 &= bx_1 - x_1^2 x_2 \end{aligned} \tag{3}$$

The above equation is called the Brusselator and is used to model oscillatory chemical reactions.

Problem 5 [25 points]. Consider the following system

$$\begin{aligned} \dot{u} &= u + v^2 \\ \dot{v} &= -v \\ \dot{w} &= -w + v^2 \end{aligned} \tag{4}$$

For this system

- Calculate the explicit solution $(u(t), v(t), w(t))$ **6 points**
- Calculate the stable and unstable subspaces for the linearization of system (4) **3 points**
- Using the explicit solution, calculate the global stable and unstable invariant manifolds **6 points**
- Calculate a global change of coordinates to bring the system to a linear form, i.e., the system is essentially a linear system masquerading as a nonlinear one. **10 points**

Hint: For the last part, the change of coordinates essentially takes the invariant manifolds of the nonlinear system to the coordinate axis of the linear system.

Problem 6 [10 points]. Consider the following system

$$
\begin{aligned}
\dot{x} &= \epsilon x - x^3 + xy \\
\dot{y} &= -y + y^2 - x^2
\end{aligned}
$$

- Describe the dynamics on the center manifold for this system near $\epsilon = 0$. **6 points**
- Draw the phase potrait on the center manifold for ϵ near 0. **4 points**

Problem 7 [10 points]. Suppose that the linearization of $\dot{x} = f(x)$ about a periodic orbit γ has a fundamental matrix solution given by:

$$\Phi(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 & 0 \\ \sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \frac{4e^{t}-e^{-2t}}{3} & \frac{2(e^{-2t}-e^{t})}{3} & 0 \\ 0 & 0 & \frac{2(e^{t}-e^{-2t})}{3} & \frac{4e^{-2t}-e^{t}}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the characteristic exponents of the periodic orbit γ.

Problem 8 [25 points total]. The following model describes the interaction of two neurons in a biological system, one of which is inhibitory and one of which is excitatory. Let x_1 be the output of the excitatory neuron and x_2 the output of the inhibitory neuron. Suppose that the evolution of x_1 and x_2 is dictated by:

$$
\begin{aligned}
x_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2) \\
x_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)
\end{aligned}
$$

where $\tau > 0$ is the characteristic time constant and $\lambda > 0$ the amplification gain and

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

- **[20 points]** Show that the system has a periodic orbit when $\lambda\tau > 1$.
- **[5 points]** Draw the phase portrait for $\tau = 1$ and $\lambda = 2$ and discuss the qualitative behavior of the system.

Problem 9 [35 points total]. The equations of motion of an n-link robot are obtained from a Lagrangian of the form:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$$

where $q \in \mathbb{R}^n$ is an n-dimensional vector of generalized coordinates representing the joint angles, $M(q)$ is a positive definite symmetric matrix and $V(q)$ is assumed to be a positive definite function of q.

The equations of motion for an n-link robot with an n-dimensional control (torque) input, F, are obtained from the Lagrange-D'Alembert principle and can be written in the form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = F(q, \dot{q})$$

where $\dot{M} - 2C$ is skew-symmetric.

- **[10 points]** With $F(q, \dot{q}) = 0$, show that the origin $(q, \dot{q}) = (0, 0)$ is stable. [Hint: Consider the Hamiltonian $H(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q)$.]
- **[10 points]** With $F(q, \dot{q}) = -K_d\dot{q}$, where K_d is a positive definite matrix, show that the origin is asymptotically stable.
- **[15 points]** With $F(q, \dot{q}) = N(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive definite matrices and q^* is a desired robot position in $\mathbb{R}^n$, show that the point $(q, \dot{q}) = (q^*, 0)$ is an asymptotically stable equilibrium point [Hint: Consider the energy function $E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q - q^*)^T K_p(q - q^*)$.]