1 Lecture: Lagrangian Mechanics

CDS 140a, Prof. Jerrold E. Marsden, November 21, 2006;
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As an aid in the introduction of the main principles governing mechanical systems first, from the Lagrangian viewpoint, through the lecture we will develop two examples, specifying one at a time all the quantities that are needed to describe the systems.

Example 1. Consider a particle moving in Euclidean 3-space, $\mathbb{R}^3$, subject to potential forces. A specific example is a planet moving around the Sun subject to its gravitational pull (assuming, for simplicity that the Sun remains stationary). As a variant of this class of examples, also consider an object whose mass is time-varying and is subject to external forces (e.g. a rocket burning fuel and generating propulsion).

Example 2. Consider a circular hoop, rotating about the $z-$axis with a certain fixed angular velocity ω, as shown in Figure 1.1.

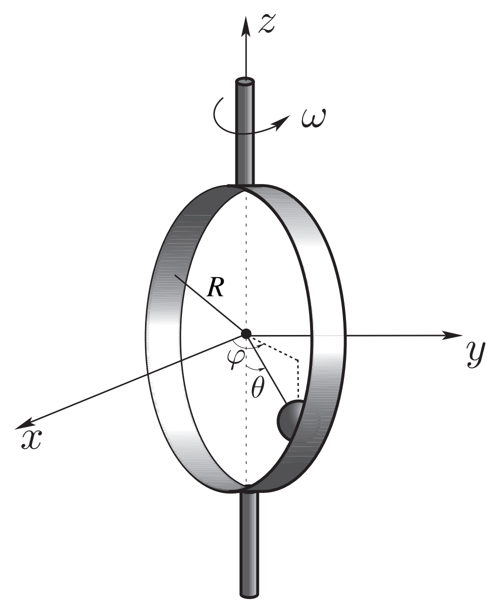

Figure 1.1: A particle moving in a rotating hoop.

Configuration Space Q. The configuration space of a mechanical system is a space whose points determine the spatial position of the system. Although this space is generally parametrized by generalized coordinates, denoted $(q^1, ..., q^n)$, it need not be an Euclidean space. It can be rather thought as a configuration *manifold*. Since we will be working with coordinates for this space, it is ok to think of Q as

Euclidean space for purposes of this lecture. However, the distinction turns out to be an important general issue.

Example 1. Here $Q = \mathbb{R}^3$ since a point in space determines where our system is; the coordinates are simply standard Euclidean coordinates: $(x, y, z) = (q^1, q^2, q^3)$.

Example 2. Here $Q = S^1$, the circle of radius R since the position of the particle is completely determined by where it is in the hoop. Note that the hoop's position in space is already determined as it has a *prescribed* angular velocity.

The Lagrangian $L(q, v)$. The ***Lagrangian*** is a function of $2n$ variables, if n is the dimension of the configuration space. These variables are the positions and velocities of the mechanical system. We write this as follows

$$L(q^1, ..., q^n,\, v^1, ..., v^n)) = L(q^1, ..., q^n,\, \dot{q}^1, ..., \dot{q}^n). \tag{1.1}$$

At this stage, the $\dot{q}^i$s are not time derivatives yet (since L is just a function of $2n$ variables, but as soon as we introduce time dependence so that the q^is are functions of time, then we will require that the v^is to be the time derivatives of the q^is.

In many (but not all) our examples we will set $L = K_E - P_E$, i.e. L is the difference between the kinetic and potential energies.

The sign in this definition is very important. For instance, consider a particle with constant mass, moving in a potential field V, which generates a force $F = -\nabla V$. We will see shortly that the equation $F = ma$ is a particular case of the Euler Lagrange (E-L) equation, therefore necessarily we need the minus sign before the P_E.

Recall also by elementary mechanics that the kinetic energy of a particle with mass m moving in $\mathbb{R}^3$ is given by $K_E = \frac{1}{2}m\|v\|^2$. This definition comes about because the kinetic energy is related to the work done by the force in a simple way as follows: if the particle governed by $F = ma$, then

$$\frac{d}{dt}\frac{1}{2}m\|v\|^2 = mv \cdot F$$

and so the fundamental theorem of calculus shows that change in its kinetic energy from point a to point b is given by:

$$\Delta K_E = \int_a^b F(t) \cdot v dt,$$

that is, the line integral of F along the path taken by the particle, or the ***work*** done by F along the path of the particle. In particular, if the force is given by $F = -\nabla V$ for a potential V, then

$$\Delta K_E = \int_a^b F(s) \cdot ds = V(a) - V(b),$$

that is,

$$\Delta\left(K_E + P_E\right) = 0,$$

which gives conservation of energy. This will be also verified below when we consider the equivalence between Hamilton's variational principle and the E-L equation.

Example 1. From the above discussion, we see that in Example 1, we should have

$$L(q,v) = \frac{1}{2}m\|v\|^2 - V(q)$$

Example 2. With reference to Figure 1.2, in an inertial (or laboratory) frame, the velocity is:

$$\mathbf{v}(t) = R\dot{\theta}\mathbf{e}_\theta + \omega R\sin\theta\mathbf{e}_\phi.$$

Where the vectors $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_R$ are an orthonormal basis of vectors in $\mathbb{R}^3$, associated with spherical coordinates. Obviously, since the mass is moving inside the hoop, its component along $\mathbf{e}_R$ is zero. The component $R\dot{\theta}\mathbf{e}_\theta$ is the velocity along $\mathbf{e}_\theta$ since the distance of the mass m from the straight down position along the circle is given by $R\theta$.

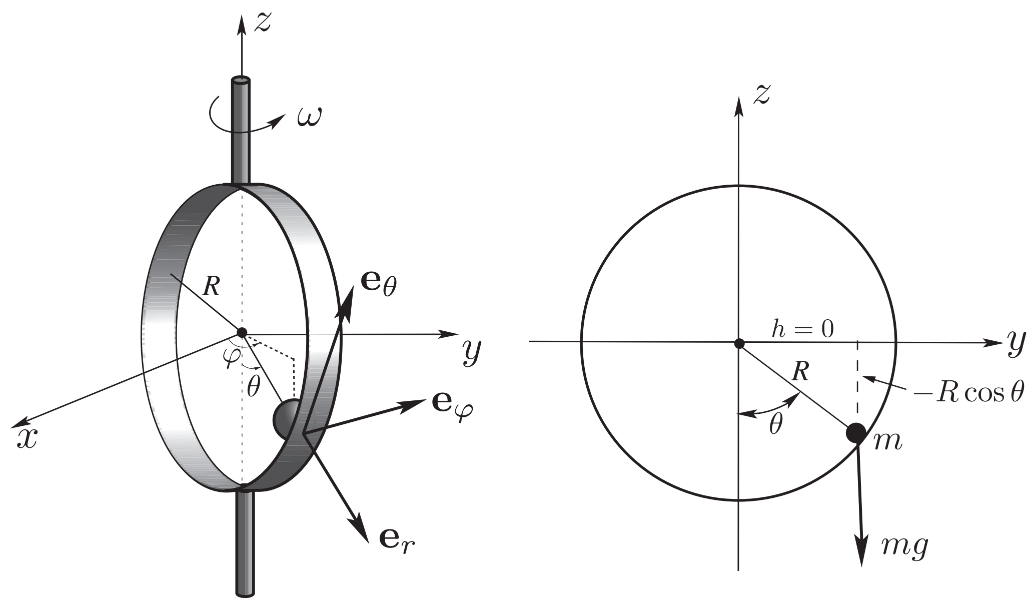

Figure 1.2: A particle moving in a rotating hoop with the attached orthonormal frame and a side view.

The kinetic energy of the particle is

$$K_E = \frac{1}{2}m\|\mathbf{v}\|^2 = \frac{1}{2}m(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta),$$

while the potential energy is given by $P_E = mgh = -mgR\cos\theta$. Therefore the Lagrangian for this system is

$$L(\theta,\dot{\theta}) = \frac{1}{2}m(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta) - mgR\cos\theta.$$

The Euler-Lagrange Equations. The first step in the description of a Lagrangian system was giving the configuration space and the second was giving the Lagrangian. Now we come to the third step, which is writing down the *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \tag{1.2}$$

Historical Note. This equation was introduced by Lagrange (Born 25 Jan 1736 in Turin, Italy, Died 10 April 1813 in Paris, France), and it corresponds to Newton's Second Law $F = ma$: being though written in generalized coordinates $(q^i, \dot{q}^i)$, the point being that a change of coordinates does not alter the form of the Euler–Lagrange equations. To this end, the definition of $L = K_E - P_E$ is crucial, and makes Lagrangians compatible and consistent with relativity theory (it is linked to covariance and independence of coordinate systems). It is interesting also to note how Lagrange originally named what we call L today as $H = K_E - P_E$, after the dutch scientist Huygens, famous and admired at that time for his works on geometric optics.

Let us go back to our examples:

Example 1. Here the Euler–Lagrange equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{d}{dt}(mv) + \nabla V = 0 \tag{1.3}$$

$$\Longrightarrow \frac{d}{dt}(mv) = -\nabla V, \tag{1.4}$$

which is the same as $F = ma$.

Example 2. Here we first compute

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta,$$

and so the Euler–Lagrange equations become

$$\frac{d}{dt}(mR^2\dot{\theta}) - (mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta) = 0.$$

That is,

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0.$$

Optional Exercise (for those who might have doubts): Write the equations of the motion of the particle in the hoop in Euclidean coordinates, and find the expression $F = ma = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, realizing that the hoop will exert forces of constraint on the particle and that one must transform the Euclidean accelerations to spherical coordinates (not a pleasant, but a straightforward task). Show that this procedure results in the same equations of motion.

Next, we will show the equivalence of Hamilton's variational principle (1830) and the E-L equation:

$$\delta \int_a^b L(q, \dot{q}) = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

The variational principle tells us that the *action* integral is stationary to perturbations of the curve going from a to b, as in Figure 1.3.

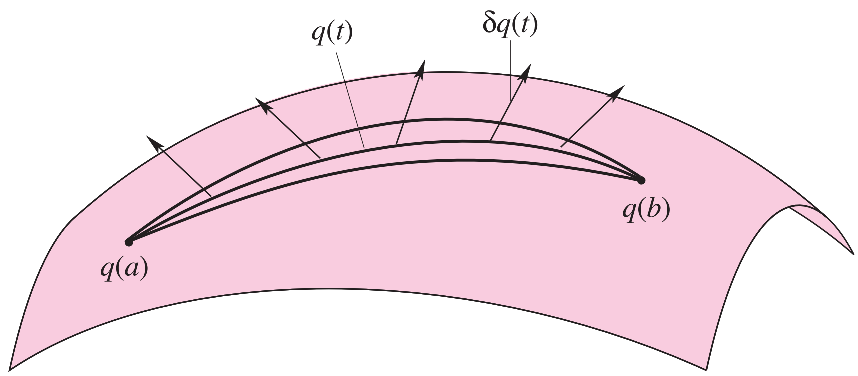

Figure 1.3: The Euler–Lagrange equations are equivalent to Hamilton's Principle: the action integral is stationary under variations of the curve $q(t)$.

What Hamilton's Principle means precisely is that the curve $q(t)$ is special; it is such that for any family of curves $q(t, \epsilon)$ satisfying the conditions

$$\begin{aligned} q(t, 0) &= q(t) \\ q(a, \epsilon) &= q(a) \\ q(b, \epsilon) &= q(b), \end{aligned}$$

we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L(q_\epsilon(t), \dot{q}_\epsilon(t)) = 0.$$

As indicated in the preceding Figure, we write:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q(t, \epsilon) = \delta q(t).$$

Using this notation, and differentiating under the integral sign and using integration by parts, Hamilton's Principle becomes

$$\begin{aligned} 0 = \frac{d}{d\epsilon}_{|\epsilon=0} \int_a^b L(q_\epsilon(t), \dot{q}_\epsilon(t)) &= \int_a^b \frac{\partial L}{\partial q}\delta q + + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\, dt \\ &= \int_a^b \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q}(\dot{\delta q})\right] dt, \end{aligned}$$

where we used equality of mixed partials in interchanging the time derivative and the ϵ derivative (that is, interchanging the overdot and the δ operations). Now integration by parts gives

$$0 = \int_a^b \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right] \delta q \, dt = 0,$$

where we there is no boundary term since δq vanishes at the endpoints because of the conditions $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$. Since this is zero for arbitrary $\delta q(t)$, the integrand must vanish. In summary, this argument shows indeed that *Hamilton's Principle is equivalent to the Euler–Lagrange equations.*

2 Lecture: Towards Hamiltonian Mechanics

CDS 140a, Prof. Jerrold E. Marsden, November 21, 2006
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External Forces. In the Last lecture we have shown that *Hamilton's Principle is equivalent to the Euler–Lagrange equations.* In the case of external forces F_{ext} we modify Hamilton's principle to the ***Lagrange-d'Alembert principle***:

$$\delta \int L(q, \dot{q}) dt + \int F_{\text{ext}} \cdot \delta q \, dt = 0$$

Note that forces that come from a potential are conventionally put into the Lagrangian. Hence, by *external* we mean forces that are not derived from potential (such as friction or the propulsion force of a rocket). However, if potential forces were to be included as external forces, then the system remains consistent.

The same argument that was uses to show that Hamilton's Principle is equivalent to the Euler–Lagrange equations shows that the Lagrange-d'Alembert principle is equivalent to the Euler–Lagrange equations with external forces:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F_{\text{ext}}(q, \dot{q})$$

For instance in the rocket example, these equations give:

$$\frac{d}{dt} (m\dot{q}) = -\nabla V + F_{\text{ext}}$$

These are the *correct equations* when the mass of the rocket is allowed to change with time. Note that one cannot just pull the m out of the derivative sign as one might guess if one naively uses $F = ma$.

For the ball in the hoop example, if we add a friction term that is proportional to the velocity (this is a bit ad hoc; in fact modeling friction realistically is a subtle business), we get

$$mR^2 \ddot{\theta} = mR^2 \omega^2 \sin\theta \cos\theta - mgR \sin\theta - \nu R \dot{\theta},$$

where we regard ν as the coefficient of friction.

The Energy Equation. Next we discuss the energy equation for a mechanical system. First suppose that there are no external forces, so that the Euler–Lagrange equations hold. Define the ***energy*** to be

$$E(q,\dot{q}) = \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L(q^i,\dot{q}^i),$$

or, for short,

$$E(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q}).$$

Using the Euler–Lagrange equations, we compute the time derivative of E with the help of the product rule and the chain rule as follows.

$$\begin{aligned}
\frac{d}{dt}E &= \sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k\right) - \sum_{j=1}^{n}\frac{\partial L}{\partial q^j}\dot{q}^j - \sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}^j}\ddot{q}^j \\
&= \sum_{k=1}^{n}\dot{q}^k\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right) \\
&= 0.
\end{aligned}$$

A similar calculation shows that when there is an external force present, then

$$\frac{d}{dt}E = F_{\text{ext}} \cdot \dot{q} = \text{ Power of external forces.}$$

For example, for the ball in the hoop, the energy is given by the expression

$$\begin{aligned}
E(\theta,\dot{\theta}) &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L(\theta,\dot{\theta}) \\
&= mR^2\dot{\theta}^2 - \left(\frac{1}{2}m(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta) - mgR\cos\theta\right) \\
&= \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}m\omega^2R^2\sin^2\theta + mgR\cos\theta.
\end{aligned}$$

Including the friction term, the energy equation becomes

$$\frac{d}{dt}E(\theta,\dot{\theta}) = -\nu R\dot{\theta}^2,$$

which, if $\nu > 0$, is strictly negative except at places where $\dot{\theta} = 0$.

The Simple Pendulum. We now look at the ball in the hoop a bit more closely. First consider the case when the hoop is not rotating (that is, $\omega = 0$) and there is no friction (that is, $\nu = 0$). In this case, the equation of motion becomes that of the ***simple pendulum***:

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$$

Note that for small oscillations, in which case $\sin\theta \approx \theta$, the equation of motion simplifies to $\ddot{\theta} + \frac{g}{R}\theta = 0$ which is a simple harmonic oscillator whose angular frequency is

$$\omega_{\text{pend}} = \sqrt{\frac{g}{R}}.$$

The phase portrait of the simple pendulum is shown in Figure 2.1.

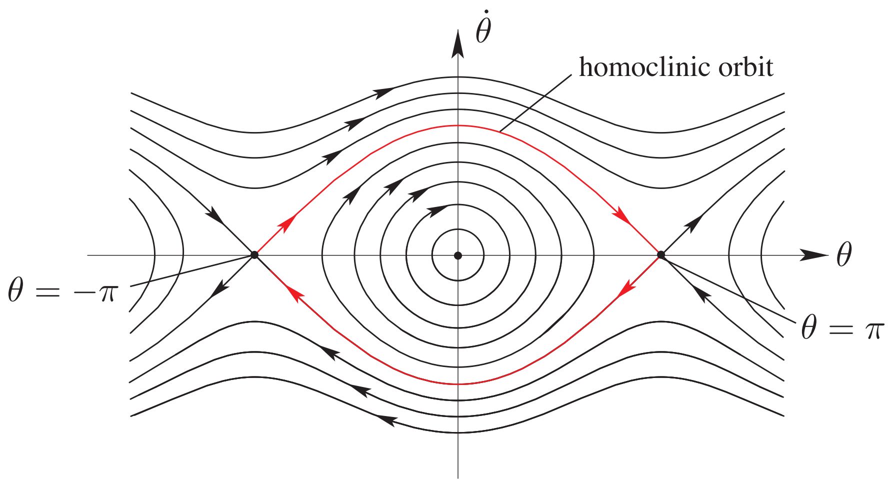


Figure 2.1: Phase portrait of the simple pendulum.

Phase Portraits for the Ball in the Hoop. We write the ball in the hoop as a first order dynamical system as follows.

$$\begin{aligned}\dot{\theta} &= v\\ \dot{v} &= \frac{g}{R}(\alpha\cos\theta - 1)\sin\theta - \beta v,\end{aligned}$$

where $\alpha = (R/g)\omega^2$ and $\beta = \nu/m$.

The equilibrium points of this system are obtained by setting $\dot{x}$ and $\dot{v}$ equal to zero. Thus, the equilibrium points correspond to zeros of $(\alpha\cos\theta - 1)\sin\theta$. If $\sin\theta = 0$, then either $\theta = 0$ or $\theta = \pi$ (plus multiples of 2π). The other equilibrium points occur when $\alpha\cos\theta = 1$. There are no other equilibria if $1/\alpha > 1$, 2 solutions if $1/\alpha < 1$ and one solution if $\alpha = 1$.

Therefore a critical value is when $\alpha = 1$; i.e., $R\omega^2 = g$. Thus, a bifurcation occurs when $\omega = \sqrt{g/R}$; i.e., interestingly, when the hoop rotates with the same angular velocity as frequency of oscillations of the *simple* pendulum. The change in the phase portrait as α crosses the critical value $\alpha = 1$ is shown in Figure 2.2. Adding a bit of dissipation does not change this picture too much from a large scale perspective, in the sense that the bifurcation from one to three equilibria still occurs at the same critical value even in the presence of dissipation, but it does turn the centers into sinks.

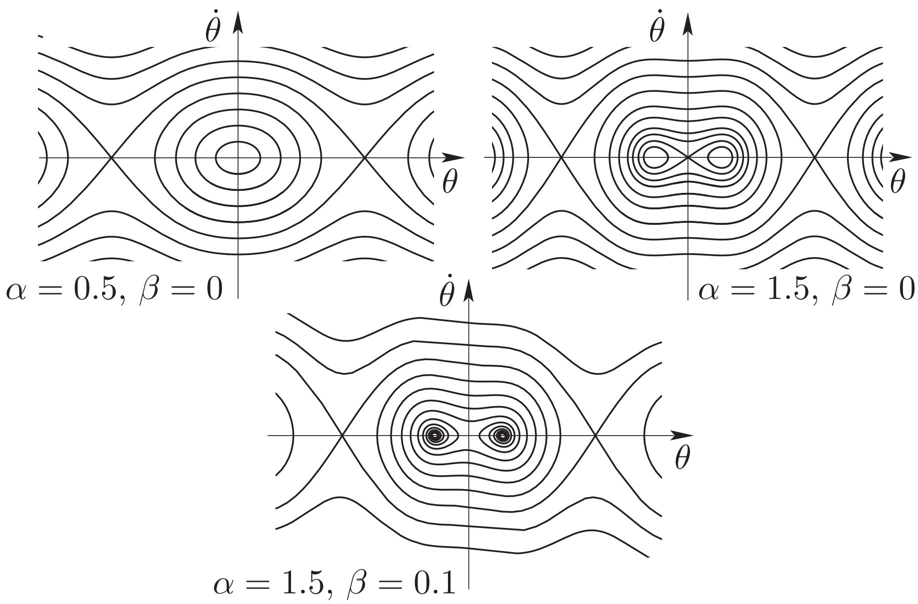


Figure 2.2: Phase portraits for the ball in the hoop. No damping, but increasing rotation rate as α increases from $\alpha = .05$ to $\alpha = 1.5$ and $\beta = 0$ and adding a bit of damping when $\beta = 0.1$.

Off-centered Hoop. So far we have considered the problem when the axis of rotation of the hoop is concentric with its axis of symmetry. When, the rotation axis is slightly shifted by an ϵ the phase portrait will be as shown in Figure 2.3. Like the symmetric case, one changes from one equilibrium inside the "pendulum homoclinic loop", but unlike the symmetric case, the one equilibrium does not splt into three, but rather two new equilibria appear through a center-saddle bifurcation as ω increases.

The Legendre Transformation. Now we show how to obtain a Hamiltonian system from a Lagrangian one. To do this, we introduce the Legendre transformation, namely defining the ***conjugate momentum*** by

$$p_i = \frac{\partial L}{\partial \dot{q}^i},$$

introduce the change of variables

$$(q^i, \dot{q}^i) \mapsto (\dot{q}^i, p_i)$$

and assume (say via the implicit function theorem) that this is a legitimate change of variables; that is, that it defines, implicitly $\dot{q}^i$ as a (smooth) function of (q^i, p_i).

Using this change of variables, introduce the Hamiltonian to be the energy but written as a function of q and p; namely, by

$$H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q^i, \dot{q}^i)$$

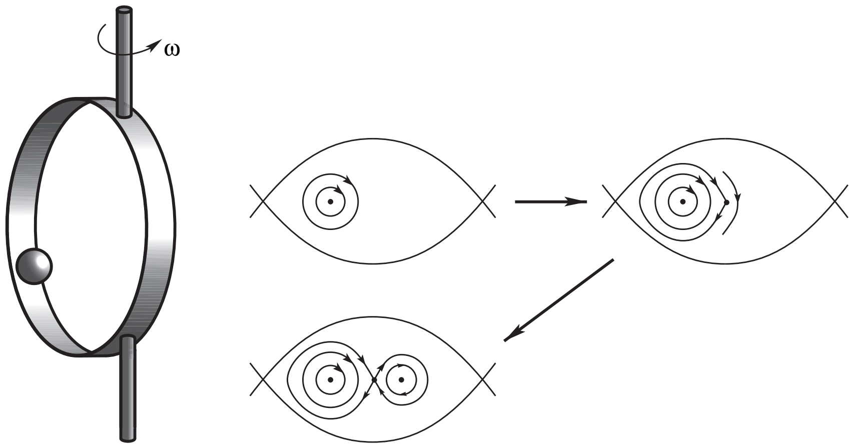


Figure 2.3: The ball in the off centered hoop and the changes in its phase portrait as the angular velocity is increases.

Given a ***Hamiltonian*** $H(q^i, p_j)$, the associated ***Hamilton equations*** are

$$\begin{aligned} \frac{d}{dt}\dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \frac{d}{dt}\dot{p}_i &= -\frac{\partial H}{\partial \dot{q}_i} \end{aligned}$$

The following theorem provides the basic link:

Theorem. *The Euler–Lagrange equations for $(q^i, \dot{q}^i)$ are equivalent to Hamilton's equations for q, p.*

Proof. First assume that the Euler–Lagrange equations hold and we will show that Hamilton's equations hold (the converse is shown in a similar manner). We compute *carefully* using the chain rule as follows:

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}^i + \sum_{j=1}^{n} \left(p_j \frac{\partial \dot{q}^i}{\partial p_j} - \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial p_i} \right) \\ &= \dot{q}^i, \end{aligned}$$

the two terms cancelling by virtue of the definition of the momentum. Similarly we calculate the other partial derivative of H as follows

$$\begin{aligned} \frac{\partial H}{\partial q^i} &= \sum_{j=1}^{n} \left(p_j \frac{\partial \dot{q}^j}{\partial q_i} - \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\delta q^i} \right) \\ &= -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \\ &= -\frac{dp_i}{dt}. \end{aligned}$$

In going from the first line to the second, we again used the definition of the conjugate momentum along with the Euler–Lagrange equations and in going to the last equality we again used the definition of the momentum. Thus we have established Hamilton's equations. QED

Example (Ball in the hoop) Recall that for the ball in the hoop,

$$L(\theta, \dot{\theta}) = \frac{1}{2} m R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos\theta$$

From this we calculate $p = \partial L / \partial \dot{\theta} = mR^2 \dot{\theta}$ and so

$$\begin{aligned} H(\theta, p) &= p\dot{\theta} - L \\ &= mR^2 \dot{\theta}^2 - L \\ &= \frac{p^2}{2mR^2} - mgR\cos\theta - \frac{1}{2} m R^2 \omega^2 \sin^2\theta \end{aligned}$$

From this Hamiltonian one can check directly that Hamilton's equations give the same equations that we had before.

Conservation of Energy. Lets have another look at conservation of energy from a Hamiltonian point of view. Assuming Hamilton's equations hold, we get, using the Chain rule,

$$\begin{aligned} \frac{d}{dt} H &= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \dot{q}^i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q^i} \right) \\ &= 0 \end{aligned}$$

Eigenvalue Theorem. Liapunov's spectral theorem deals with the important role played by the distribution of eigenvalues of the linearization of a system $\dot{x} = f(x)$ at an equilibrium point. For Hamiltonian (and also Lagrangian) systems, there is a severe restriction on how the eigenvalues can appear. In fact, the next theorem shows that the eigenvalues must be distributed symmetrically not only with respect to the real axis (always true for any real system), but, and this is very special, *also with respect to the imaginary axis.*

Theorem. *Consider a Hamiltonian $H(q^i, p_i)$ and the corresponding Hamiltonian dynamical system:*

$$\begin{aligned} \frac{d}{dt} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \frac{d}{dt} \dot{p}_i &= -\frac{\partial H}{\partial \dot{q}_i} \end{aligned}$$

Let $(\overline{q}^i, \overline{p}_i)$ be an equilibrium point; i.e., a critical point of H. Then,the spectrum of the linearization at $(\overline{q}^i, \overline{p}_i)$ is symmetric with respect to the imaginary axis.

For example, this means that in a Hamiltonian system, the spectrum at an equilibrium point can never be totally in the left half plane. The student should calculate the eigenvalues for the ball in the rotating hoop to verify that this symmetry does hold in that case.

Proof. Let $z = (q^i, p_i)$ and write Hamilton's equations in matrix form as follows

$$\dot{z} = JDH(z)$$

where

$$\dot{z} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

and the $2n \times 2n$ matrix J, the *symplectic matrix*, is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the "ones" stand for the $n \times n$ identity matrix. Also, write the derivative of H as

$$DH(z) = \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \\ \frac{\partial H}{\partial p^i} \end{pmatrix}$$

Note these easily verified properties of J: First of all, $J^T J = I$ (that is, J is an orthogonal matrix; in particular, in the case $n = 1$, it is a rotation by 90^o). Second, $J^2 = -1$. Thirdly, note that $\det J = 1$.

The linearization of the equations $\dot{z} = JDH(z)$ at a critical point $\overline{z} = (\overline{q}, \overline{p})$ is given by

$$\dot{z} = JD^2H(\overline{z})z =: Az,$$

where $A = JD^2H(\overline{z})$ and where $D^2H(\overline{z})$ is the matrix of second partial derivatives of H evaluated at the equilibrium point $\overline{z}$.

Next, note that

$$\begin{aligned} A^T J + JA &= D^2H(\overline{z})J^T J + JJD^2H(\overline{z}) \\ &= D^2H(\overline{z}) - D^2H(\overline{z}) = 0 \end{aligned}$$

where we have used the fact that $D^2H(\overline{z})$ is symmetric due to equality of mixed partials, $J^T J = I$ and $J^2 = -I$.

Now the characteristic polynomial p of the matrix A is given by

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det(J(A - \lambda I)J) \\ &= \det(JAJ + \lambda I) \end{aligned}$$

where in going from the first line to the second, we used the fact that $\det J = 1$ and in going from the second line to the third we used the fact that $J^2 = -I$. However, from our calculation above, $JA = -A^T J$ and so $JAJ = A^T$. Thus, $\det(JAJ + \lambda I) = \det(A^T + \lambda I) = \det(A + \lambda I) = p(-\lambda)$. Thus, $p(\lambda) = p(-\lambda)$. QED

Homoclinic Orbit of the Simple Pendulum. We return to the simple pendulum. Here our goal is to calculate the equation of the hetroclinic orbit. In general the trajectories can be found using elliptic functions. However, it is interesting that the homoclinic orbit can be written in terms of elementary functions. Here is the procedure:

First of all, recall that the equations (with $g/R = 1$) are

$$\ddot{\theta} + \sin\theta = 0$$

which have the energy integral

$$E = \frac{1}{2}\dot{\theta}^2 - \cos\theta$$

which we can rewrite as

$$\dot{\theta} = \sqrt{2(E + \cos\theta)}$$

and so

$$\int \frac{d\theta}{\sqrt{2(E + \cos\theta)}} = \int dt = t + c$$

The value of E on the homoclinic trajectory equals the value of E at the saddle point $(\pi, 0)$, namely $E = 1$; in this case, the integral is available (eg, in tables of integrals) and is given by

$$\int \frac{d\theta}{\sqrt{2(1 + \cos\theta)}} = \frac{1}{2}\log\left(\frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)}\right)$$

We can also fix the integration constant by choosing $t = 0$ to correspond to the point $\theta = 0$ (note that the point $(0, 1)$, which has energy $E = 1$ lies on the homoclinic orbit); from these calculations, one concludes that the homoclinic orbit is given by

$$\theta(t) = \pm 2\tan^{-1}(\sinh t).$$

Exercises

1. Derive the equations of motion for a particle in a circular hoop spinning with angular velocity ω, as in these notes, but about a line that is a distance ϵ off center. What can you say about the equilibria as functions of ϵ and ω?

2. Consider Duffing's equation

$$\ddot{x} - \beta x + \alpha x^3 = 0,$$

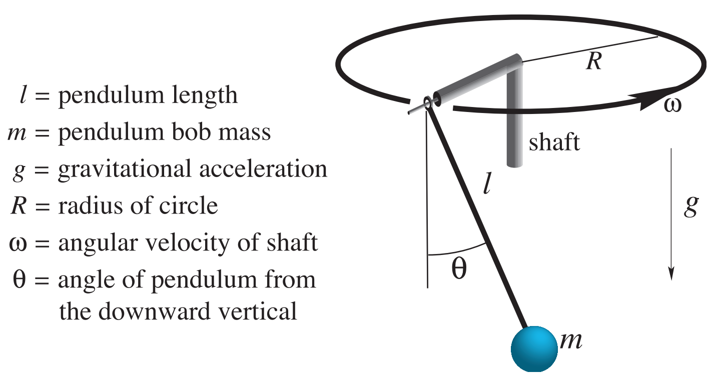
where α and β are positive constants. Derive a formula for the two homoclinic orbits for this equation.

3. Determine the equations of motion for a ball in a light rotating hoop—this time the hoop is not forced to rotate with constant *angular velocity*, but rather is free to rotate (its *angular momentum* μ will then be conserved).

4. Consider the pendulum shown in the following figure. It is a planar pendulum whose suspension point is being whirled in a circle with constant angular velocity ω by means of a vertical shaft, as shown. The plane of the pendulum is orthogonal to the radial arm of length R. Ignore frictional effects.

   (a) Using the notation in the figure, find the equations of motion of the pendulum.

   (b) Regarding ω as a parameter, examine the bifurcation of equilibria that occurs as the angular velocity ω of the shaft is increased.



5. Show that the Euler–Lagrange equations as well as the Legendre transformation are consequences of the *Hamilton-Pontryagin Principle*:

$$\delta \int_a^b \left[L(q, v) - p \cdot (\dot{q} - v) \right] dt = 0$$

where q, v and p can all be varied independently.

6. Consider the mechanical system shown in the Figure below. The mass M slides (without friction) along the curve $y = x^2$. The mass m hangs by a light rod (as a planar pendulum) from the mass M, and makes an angle θ with the vertical, as shown. Both masses are subject to a downward gravitational force.

   (a) Write down a Lagrangian for the system.

   (b) State the general Euler–Lagrange equations for a system with a Lagrangian $L(x, \theta, \dot{x}, \dot{\theta})$.

   (c) Does this system have a conserved energy? If so, compute it.

   (d) When M is stationary at $x = 0$ and m is hanging vertically (with $\theta = 0$), one has an equilibrium. Is it stable? Asymptotically stable?