Instructions

- This is a closed book 90 minute exam consisting of 3 problems.
- Please begin the solution of each problem on a separate page.
- For Problem 2, please provide a proof if you think the answer is “True” and a counterexample otherwise. Getting the answer correct without supporting materials (proof or counterexample) will only earn you 2 out of 7 points. There is no penalty for incorrect answers.
Problem 1 (30 points) Please solve each of the following short computational problems.

(a) Compute the $\infty$-norm of transfer function $(a > 0)$:

$$\hat{G}(s) = \frac{e^{-s}}{s + a}$$

(b) Find the $\mathcal{H}_2$ norm of the system described by:

$$\dot{y} + y = u$$

(c) The input to a system with transfer function

$$\hat{G}(s) = \frac{1}{(s + 1)^2}$$

has amplitude not exceeding 1 for all $t \geq 0$. What is the upper bound on the amplitude of the output signal?

(d) Find a state-space representation for the system described by the following transfer matrix:

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{s-1} & \frac{1}{s+2} \end{bmatrix}$$

For parts (e) and (f), consider the LTI system

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = [1]$$

(e) Find $x(t)$ and $y(t)$, $t \geq 0$, for the autonomous system with initial condition

$$x(0) = \begin{bmatrix} 1 \\
2 \\
1 \end{bmatrix}$$

!A, B, C and D should not appear explicitly in your answer!

(f) Compute the transfer matrix of the system.
Problem 2 (35 points) TRUE or FALSE? If the answer is TRUE, please provide a sketch of the proof. If the answer is FALSE, please provide a counterexample.

(a) Consider a system \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with transfer matrix \( \hat{G}(s) \). Each eigenvalue of \( A \) is a pole of \( \hat{G}(s) \).

(b) If \( \hat{G}(s) \) is a stable rational transfer function and \( \hat{G}_1(s) = \hat{G}(rs) \), then \( \|\hat{G}_1\|_2 = r^{-\frac{1}{2}} \|\hat{G}\|_2 \) for any \( r > 0 \).

(c) Given a matrix \( M \in \mathbb{C}^{n \times n} \), there exists a diagonal matrix \( D \) such that

\[
\|DMD^{-1}\|_2 < 1
\]

iff the LMI

\[
\begin{bmatrix}
-X & 0 \\
0 & M^*XM + X
\end{bmatrix} < 0
\]

is feasible.

(d) Consider \( \hat{G}_1(s) = \frac{1}{s + a_1} \) and \( \hat{G}_2(s) = \frac{1}{s + a_2} \) where \( a_1 > 0, a_2 > 0 \).

\[
\|\hat{G}_2\hat{G}_1\|_2 \leq \|\hat{G}_1\|_2\|\hat{G}_2\|_2
\]

whenever \( a_1 + a_2 > 2 \).

(e) For \( A \in \mathbb{C}^{n \times n} \),

\[
\min_{\Delta \in \mathbb{C}^{n \times n}} \{ \|\Delta\|_2 | I - \Delta A \text{ is singular } \} = \frac{1}{\sigma_{\max}(A)}
\]
Problem 3 (35 points: (a)-8 points, (b)-12 points, (c)-15 points)

Consider the dynamical system with input $u$ and output $y$ given by:

$$\ddot{y} + ay + by + cy^2 = u + \dot{u}$$

where $a$, $b$ and $c$ are some scalar parameters.

(a) Find a second order state space realization of the system?

For parts (b) and (c) set the values of the parameters to be: $a = 3$, $b = c = 2$.

(b) Show that the origin is an asymptotically stable equilibrium point. Is stability local or global?

(c) Find a neighborhood of the origin (i.e. a ball of non-zero radius) such that every trajectory starting in this neighborhood converges to the origin as $t \to \infty$? This is called a “region of attraction” of the equilibrium point at 0.