CDS 212 - Midterm Exam

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Instructions

- This is a closed book 90 minute exam consisting of 3 problems.
- Please begin the solution of each problem on a *separate* page.
- For Problem 2, please provide a proof if you think the answer is "True" and a counterexample otherwise. Getting the answer correct without supporting materials (proof or counterexample) will only earn you 2 out of 7 points. There is no penalty for incorrect answers.

Problem 1 ***(30 points)*** Please solve each of the following *short computational* problems.

(a) Compute the ∞-norm of transfer function ($a > 0$):

$$\hat{G}(s) = \frac{e^{-s}}{s+a}$$

(b) Find the $\mathcal{H}_2$ norm of the system described by:

$$\dot{y} + y = u$$

(c) The input to a system with transfer function

$$\hat{G}(s) = \frac{1}{(s+1)^2}$$

has amplitude not exceeding 1 for all $t \geq 0$. What is the upper bound on the amplitude of the output signal?

(d) Find a state-space representation for the system described by the following transfer matrix:

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{s-1} & \frac{1}{s+2} \end{bmatrix}$$

For parts (e) and (f), consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; D = [1]$$

(e) Find $x(t)$ and $y(t)$, $t \geq 0$, for the autonomous system with initial condition

$$x(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

!A, B, C and D should not appear explicitly in your answer!

(f) Compute the transfer matrix of the system.

Problem 2 ***(35 points)*** TRUE or FALSE? If the answer is TRUE, please provide a sketch of the proof. If the answer is FALSE, please provide a counterexample.

(a) Consider a system $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with transfer matrix $\hat{G}(s)$. Each eigenvalue of A is a pole of $\hat{G}(s)$.

(b) If $\hat{G}(s)$ is a stable rational transfer function and $\hat{G}_1(s) = \hat{G}(rs)$, then $\|\hat{G}_1\|_2 = r^{-\frac{1}{2}}\|\hat{G}\|_2$ for any $r > 0$.

(c) Given a matrix $M \in \mathbb{C}^{n\times n}$, there exists a diagonal matrix D such that

$$\|DMD^{-1}\|_2 < 1$$

iff the LMI

$$\begin{bmatrix} -X & 0 \\ 0 & M^*XM + X \end{bmatrix} < 0$$

is feasible.

(d) Consider $\hat{G}_1(s) = \dfrac{1}{s+a_1}$ and $\hat{G}_2(s) = \dfrac{1}{s+a_2}$ where $a_1 > 0$, $a_2 > 0$.

$$\|\hat{G}_2\hat{G}_1\|_2 \leq \|\hat{G}_1\|_2\|\hat{G}_2\|_2$$

whenever $a_1 + a_2 > 2$.

(e) For $A \in \mathbb{C}^{n\times n}$,

$$\min_{\Delta\in\mathbb{C}^{n\times n}} \{\|\Delta\|_2 | I - \Delta A \text{ is singular } \} = \frac{1}{\sigma_{\max}(A)}$$

Problem 3 ***(35 points: (a)-8 points, (b)-12 points, (c)-15 points)***

Consider the dynamical system with input u and output y given by:

$$\ddot{y} + a\dot{y} + by + cy^2 = u + \dot{u}$$

where a, b and c are some scalar parameters.

(a) Find a second order state space realization of the system?

For parts (b) and (c) set the values of the parameters to be: $a = 3$, $b = c = 2$.

(b) Show that the origin is an asymptotically stable equilibrium point. Is stability local or global?

(c) Find a neighborhood of the origin (i.e. a ball of non-zero radius) such that every trajectory starting in this neighborhood converges to the origin as $t \to \infty$? This is called a "region of attraction" of the equilibrium point at 0.