CDS 212 - Homework #2

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Posted on: Sunday October 21, 2007
Due on: Tuesday October 30, 2007
Remark: One again, please note down on your solution the amount of time you spent on this homework set.

Problem 1
Solve the following problems in Doyle, Francis and Tannenbaum:

(a) Problem 2.2

(b) Problem 2.4

(c) Problem 2.11

(d) Problem 2.13

Problem 2
Consider the following three systems:
System S_1:

$$\begin{aligned}\dot{x}_1 &= -x_1 + (x_1 + x_2)^3 \\ \dot{x}_2 &= -x_2 - (x_1 + x_2)^3\end{aligned}$$

System S_2:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2)\end{aligned}$$

System S_3:

$$\begin{aligned}\dot{x}_1 &= x_2|x_1| \\ \dot{x}_2 &= -x_1|x_2|\end{aligned}$$

and the following functions $V_i : \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned} V_a(x_1, x_2) &= x_1^2 + x_2^2 \\ V_b(x_1, x_2) &= |x_1| + |x_2| \\ V_c(x_1, x_2) &= \max\{|x_1|, |x_2|\} \end{aligned}$$

(a) What are the equilibrium points of each of these three systems?

(b) We are interested in analyzing stability of the equilibrium point(s) of S_1. Which of the given functions, if any, is a valid Lyapunov function?

Please explain your answers.

Problem 3
The objective of this problem is to understand Lyapunov methods for discrete-time LTI systems, by building on what we learned in class for continuous-time systems. Consider a discrete-time LTI system:

$$x(t+1) = Ax(t)$$

and consider the function $V = x'Px$ where P is a symmetric positive definite matrix.

(a) Under what conditions is $V(x)$ a Lyapunov function for this system?

(b) Under what conditions does $V(x)$ allow us to prove asymptotic stability of the system?

Express your answers in terms of solutions of suitable Lyapunov equations.

Problem 4
In this problem, we wish to study the stability of an nth order LTI system of the form:

$$\dot{x}(t) = (A + \Delta)x(t)$$

where Δ is a real perturbation. In particular, we wish to find a good lower bound on the size of the smallest perturbation that will destabilize the system. Assume that the given matrix A is Hurwitz, and define the stability margin of the system as:

$$\gamma(A) = \min_{\Delta \in \mathbb{R}^{n \times n}} \{\|\Delta\|_2 | A + \Delta \text{ is not Hurwitz }\}$$

(a) Consider the Lyapunov equation $A'P + PA = -Q$ and show that $\frac{\sigma_{min}(Q)}{2\sigma_{max}(P)}$ is a lower bound for $\gamma(A)$.

(b) An alternative way of computing a lower bound for $\gamma(A)$ consists of dropping the requirement that Δ is real. In particular, show that:

$$\min_{\Delta \in \mathbb{C}^{n \times n}} \{\|\Delta\|_2 | A + \Delta \text{ is not Hurwitz }\} = \min_{w \in \mathbb{R}} \sigma_{min}(A - jwI)$$

Why does this give us a lower bound on $\gamma(A)$?

(c) We can improve on the bound above by using the information that Δ is in fact real. Define

$$A_w = \begin{bmatrix} A & wI \\ -wI & A \end{bmatrix}.$$

Show that:

$$\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{min}(A_w).$$

Problem 5

One way of extending the concept of p-stability to nonlinear systems with an equilibrium point at the origin is by restricting the allowable inputs and assuming the system is initialized at 0, effectively restricting the analysis to a local region around the origin. In this case, a system $\mathcal{H}$ is said to be locally p-stable if there exists $C_o > 0$ such that:

$$\|\mathcal{H}(u)\|_p \leq C_o \|u\|_p$$

for all u with $\|u\|_p \leq \delta$. Consider the nonlinear system given by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \\ y &= g(z) \end{aligned}$$

where C and D are assumed to be row matrices (i.e. the system is multi-input single-output) and A is assumed to be Hurwitz.

(a) When $g(x) = cos(x)$, is the system 1-stable? 2-stable? ∞-stable? Explain your answers.

(b) When $g(x) = \begin{cases} x & |x| \leq 1 \\ 1 & |x| \geq 1 \end{cases}$, is the system 1-stable? 2-stable? ∞-stable? Explain your answers.

(c) When $g(x) = sin(x)$, is the system 1-stable? 2-stable? ∞-stable? Explain your answers.