CDS 212 - Final Exam

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INSTRUCTIONS : Please read carefully!

(1) Description & duration of the exam:
◇ The exam consists of 6 problems.
◇ You have a total of **24 consecutive hours** to work on the exam. The clock starts ticking when you first read these instructions.
◇ All exams need to be returned back to me by **Monday, December 10 at 9:00am -** no exceptions will be made.
◇ Please hand in the exam at my office as soon as possible after your time is up. You may hand it to me in person, or to either of my office mates. Please do not leave it outside the door!

(2) Exam policies:
◇ You are to abide by the Caltech honor code.
◇ You may not discuss this exam with anyone.
◇ This is an open book exam. You may use any of the following resources:
⋆ Your *own* class notes, taken during lectures.
⋆ The two required textbooks, "Feedback Control Theory" by Doyle, Francis and Tannenbaum and "A Course in Robust Control Theory" by Dullerud & Paganini.
⋆ Your own solutions to the homeworks and the midterm exam, as well as the solutions provided by me.
◇ Calculators and computers are NOT allowed.
◇ You may NOT consult any other textbooks or references.
◇ You may NOT use the class notes of any other class you have taken or are currently taking, or the CDS 212 class notes or material from previous quarters.
◇ You may NOT consult the Internet.

(3) Additional comments:
◇ Please begin the solution of each problem on a separate page.
◇ Please be neat and clearly label your work.

GOOD LUCK!

Problem 1 ***(20 points)*** Please solve each of the following problems.

(a) Find the $\mathcal{H}_2$ norm of the system with transfer function

$$G(s) = \frac{1}{s^3 + 2s^2 + s + 1}$$

(b) Find a minimal state space realization for the transfer matrix

$$H(s) = \begin{bmatrix} 1/s & 1/s \\ 0 & 1/s \end{bmatrix}$$

(c) Compute the poles and zeroes of the following transfer matrix:

$$G(s) = \begin{bmatrix} \frac{s-1}{s-2} & \frac{2}{s+1} \\ \frac{s}{s+1} & 0 \end{bmatrix}$$

(d) Verify whether the following system is unstable, stable i.s.L. or asymptotically stable?

$$\begin{aligned} \dot{x}_1 &= -2x_1x_2^2 \\ \dot{x}_2 &= -2x_1^2x_2 - 4x_2^3 \end{aligned}$$

Problem 2 ***(20 points)*** TRUE OR FALSE? If your answer is "FALSE", please give a counterexample. If your answer is "TRUE", please provide a proof. You will earn 1 point for each correct answer, and 4 more points for each correct proof or counterexample. There is no penalty for incorrect answers, so when in doubt give it your best educated guess. No partial credit will be given for partially correct proofs.

(a) Consider an open loop plant $P(s) = \frac{(s-1)(s-3)}{s(s-2)(s-4)}$. The *minimal* possible value of the closed loop sensitivity logarithmic integral

$$\int_0^\infty log|S(jw)|dw$$

in the standard unity feedback setup (i.e. setup in Figure 5.1 of DFT) is 12π.

(b) The $\mathcal{L}_2$-induced norm of the system with transfer function

$$G(s) = \frac{1}{s-2}$$

is $\frac{1}{2}$.

(c) The system with transfer function

$$H(s) = \frac{1}{s + 0.5 + e^{-st}}$$

is stable for any $t \in [0, 1)$.
Hint: $|e^{-jw} - 1| \leq |w|$ *for all real frequencies* w.

(d) If $\sigma_{\max}(A) < 1$ and A^{-1} exists, then:

$$\sigma_{\max}[A(I-A)^{-1}] \leq \frac{1}{\sigma_{\min}[A^{-1}] - 1}$$

Problem 3 ***(20 points)***

A SISO open loop plant has a zero at $s = 2$ and a pole at $s = 3$. Find the best lower bound you can on the $\mathcal{H}_\infty$ norm of the closed loop complementary sensitivity function T, if it is known that the closed loop system is stable and that $|S(jw)| \leq 0.1$ for $|w| < 4$.

Problem 4 ***(8 points)***

Consider an n-dimensional discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

and assume that the system is reachable: That is, the input can always be chosen to take the system from the origin to any specified state $x_o \in \mathbb{R}^n$ in at most n steps.

Now suppose that $u(k)$ is generated according to a nonlinear feedback scheme given by

$$u(k) = w(k) + f(x(k))$$

where $f(.)$ is an arbitrary but known function, and w is the new control input for the closed loop system. Show that $w(k)$ can always be chosen to take the state of the closed loop system from the origin to any specified target state in at most n steps, thus proving that reachability is preserved under arbitrary state feedback.

Problem 5 ***(12 points)***

Consider an n-th order discrete-time (DT) system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

We will say that the system is “observable” if the initial state can be uniquely determined by observing the system’s inputs and outputs over at most n time steps.

(a) Consider an observable DT n-th order system and suppose that:

$$\begin{aligned} D &= 0 \\ CB &= 0 \\ CAB &= 0 \\ &\vdots \\ CA^{n-2}B &= 0 \end{aligned}$$

Show that under these conditions, we can uniquely determine the initial state of the system from *output measurements alone*.

(b) Show that the sufficient condition in part (a) is also necessary when the output y is a scalar.

(c) Verify that for SISO systems, the condition in (a) corresponds to the transfer function having no (finite) zeroes.

Problem 6 ***(20 points)***

Let $N_o(s)$ and $D_o(s)$ be polynomials in s, and let $N_\delta(s)$ and $D_\delta(s)$ be n-dimensional row vectors whose entries are polynomials in s. Consider the set of SISO plants given by:

$$\mathcal{P} = \left\{ \frac{N_o(s) + N_\delta(s)\delta}{D_o(s) + D_\delta(s)\delta} \mid \delta \in \mathbb{R}^n, |\delta_i| \leq \gamma \right\}$$

Let C be a controller that stabilizes the nominal plant

$$P_o(s) = \frac{N_o(s)}{D_o(s)}$$

in the unity feedback setup depicted in Figure 5.1 of DFT. Compute the *largest* γ such that the closed loop system is robustly stable.