Lecture 8

17. Stability Analysis

For an n – dimensional vector $x \in R^n$, the L_2 and the L_∞ *vector* norms are defined as:

$$\begin{aligned} \|x\| &= \sqrt{x^T x} \\ \|x\|_\infty &= \max_{1 \le i \le n} |x_i| \end{aligned} \tag{17.1}$$

It is easy to show that these two norms satisfy the following relation:

$$\|x\|_\infty \le \|x\| \le \sqrt{n}\,\|x\|_\infty \tag{17.2}$$

If a vector x is time-dependent, then its *truncated* L_∞ *functional* norms is defined as:

$$\|x(t)\|_{L_\infty} = \max_{t_0 \le \tau \le t} \|x(\tau)\| \tag{17.3}$$

Note that in (17.3), the L_2 vector norm is used instead of the L_∞ vector norm. For a matrix A, vector induced L_2 and L_∞ matrix norms will be denoted by $\|A\|$ and $\|A\|_\infty$, correspondingly. Suppose that

$$\begin{aligned} K_x^{\max} &= \max_{t \ge 0} \left\| \hat{K}_x(t) \right\| \\ \Lambda^{\max} &= \max_{t \ge 0} \left\| \hat{\Lambda}(t) \right\| \end{aligned} \tag{17.4}$$

are the parameter bounds for the Projection Operator used in the adaptive laws (16.19).

First, we state and prove the following fact about input-to-state stability of a linear time-invariant (LTI) system.

Lemma 17.1
There exist strictly positive constants (k_A, λ_A), such that starting at any initial condition $x(0) = x_0$, the state vector $x(t) \in R^n$ of the LTI system

$$\dot{x} = Ax + Bu \tag{17.5}$$

with a Hurwitz matrix A and an external input $u(t) \in R^m$, satisfies

$$\left\|x(t)\right\|_{L_\infty} \le k_A \left(\left\|x_0\right\| + \frac{\left\|B\right\|}{\lambda_A} \left\|u(t)\right\|_{L_\infty} \right) \tag{17.6}$$

<u>*Proof*</u>: Since A is Hurwitz then there must exist strictly positive constants (k_A, λ_A) such that $\left\|e^{tA}\right\| \le k_A e^{-t\lambda_A}$. In this case, truncated L_∞ norm upper bound for the solution of (17.5) can be found.

$$\begin{aligned}
\left\|x(t)\right\|_{L_\infty} &= \max_{0\le s\le t} \left\| e^{sA_{ref}} x_0 + \int_0^s e^{(t-\tau)A_{ref}} B u(\tau) d\tau \right\| \\
&\le \max_{0\le s\le t} \left(k_A e^{-\lambda_A s} \left\|x_0\right\| + k_A \left\|B\right\| \int_0^s e^{-\lambda_A (t-\tau)} \left\|u(\tau)\right\| d\tau \right) \\
&\le k_A \left(\left\|x_0\right\| + \left\|B\right\| \int_0^t e^{-\lambda_A (t-\tau)} \left\|u(\tau)\right\| d\tau \right) \le k_A \left(\left\|x_0\right\| + \left\|B\right\| \int_0^t e^{-\lambda_A (t-\tau)} d\tau \max_{0\le\tau\le t} \left\|u(\tau)\right\| \right) \\
&\le k_A \left(\left\|x_0\right\| + \frac{\left\|B\right\|}{\lambda_A} \left\|u(\tau)\right\|_{L_\infty} \right)
\end{aligned} \tag{17.7}$$

This completes the proof.

Next, we find *sufficient* conditions to guarantee uniform ultimate boundedness (UUB) of all the signals in the system (15.1), which is controlled by a filtered MRAC controller, in the form of (16.5), (16.10) and (16.19). Towards that end, introduce the so-called *ideal reference model*

$$\dot{x}_{ref}^* = A_{ref} x_{ref}^* + B_c r(t) \tag{17.8}$$

and let

$$e_{ref} = x_{ref} - x_{ref}^* \tag{17.9}$$

represent the error between the actual reference model (16.10) and its ideal target (17.8). Then the reference model error dynamics can be computed as:

$$\dot{e}_{ref} = A_{ref} e_{ref} + B \hat{\Lambda} \left(G(s) - I_{m\times m} \right) \underbrace{\hat{K}_x^T x}_{u_c} \tag{17.10}$$

Using (16.11) and (17.9), gives

$$\dot{e}_{ref} = A_{ref} e_{ref} + B \hat{\Lambda} \left(G(s) - I_{m\times m} \right) \hat{K}_x^T \left(e_{ref} + x_{ref}^* + e \right) \tag{17.11}$$

where $\hat{\Lambda}$, $\hat{K}_x$, x_{ref}^* and e are UUB. We will use Small Gain Theorem type arguments to find sufficient conditions for input-to-state stability of (17.11). This system can be

represented by the two feedback-connected systems H_1 and H_2, as shown in the block-diagram:

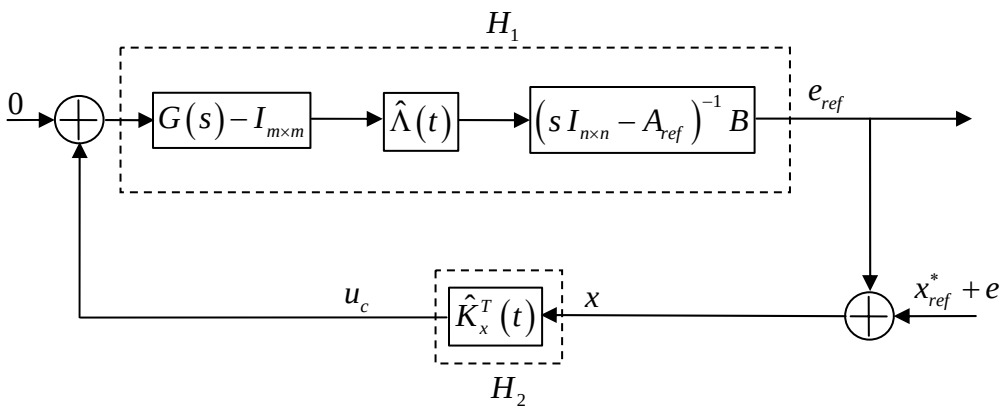

Figure 1: Reference Model Error Dynamics

According to the figure, these two systems are defined as

$$e_{ref} = H_1 u_c \Leftrightarrow \dot{e}_{ref} = A_{ref}\, e_{ref} + B\,\hat{\Lambda}\left(G(s)-I_{m\times m}\right)u_c \tag{17.12}$$

and

$$u_c = H_2\, x \Leftrightarrow u_c = \hat{K}_x^T\left(e_{ref} + x_{ref}^* + e\right) \tag{17.13}$$

Note that H_1 is driven by the commanded control input

$$u_c = \hat{K}_x^T\left(e_{ref} + x_{ref}^* + e\right) \tag{17.14}$$

while H_2 is driven by the system state vector

$$x = e_{ref} + \underbrace{\left(x_{ref}^* + e\right)}_{\text{Bounded}} \tag{17.15}$$

Let γ_{H_1} denote the L_1 gain of the system H_1. One can easily show that the gain value is finite. In fact, applying Lemma 17.1 to (17.12) gives

$$
\begin{aligned}
\left\|e_{ref}(t)\right\|_{L_\infty} &\le k_{A_{ref}}\left(\left\|e_{ref}(0)\right\| + \frac{\|B\|\Lambda^{\max}}{\lambda_{A_{ref}}}\left\|\left(G(s)-I_{m\times m}\right)u_c\right\|_{L_\infty}\right) \\
&\le k_{A_{ref}}\left(\left\|e_{ref}(0)\right\| + \frac{\|B\|\Lambda^{\max}}{\lambda_{A_{ref}}}\left(\underbrace{\left\|G(s)u_c\right\|_{L_\infty}}_{\le \gamma_G \|u_c\|_{L_\infty}} + \left\|u_c\right\|_{L_\infty}\right)\right) \\
&\le k_{A_{ref}}\left\|e_{ref}(0)\right\| + \frac{k_{A_{ref}}}{\lambda_{A_{ref}}}\|B\|\Lambda^{\max}\left(\gamma_G+1\right)\left\|u_c\right\|_{L_\infty}
\end{aligned}
\tag{17.16}
$$

where γ_G is the L_1 gain of the proper and stable filter $G(s)$. The inequality in (17.16) gives finite upper bound for the L_1 gain γ_{H_1} of the system H_1:

$$
\gamma_{H_1} \le \frac{k_{A_{ref}}}{\lambda_{A_{ref}}}\|B\|\Lambda^{\max}\left(\gamma_G+1\right) \tag{17.17}
$$

Furthermore, the relations

$$
\begin{aligned}
\left\|u_c\right\|_{L_\infty} &= \left\|\hat{K}_x^T\left(e_{ref}+x_{ref}^*+e\right)\right\|_{L_\infty} \le K_x^{\max}\left(\left\|e_{ref}(t)\right\|_{L_\infty} + \underbrace{\left\|x_{ref}^*(t)\right\|_{L_\infty}}_{\le x_{ref\,\max}^*} + \underbrace{\left\|e(t)\right\|_{L_\infty}}_{\le e_{\max}}\right) \\
&\le K_x^{\max}\left\|e_{ref}(t)\right\|_{L_\infty} + K_x^{\max}\left(x_{ref\,\max}^* + e_{\max}\right)
\end{aligned}
\tag{17.18}
$$

imply that the L_1 gain γ_{H_2} of the system H_2 is also finite:

$$
\gamma_{H_2} \le K_x^{\max} \tag{17.19}
$$

Finally, using Small Gain Theorem allows to claim input-to-state stability of the system (17.11), if

$$
\gamma_{H_1}\gamma_{H_2} < 1 \tag{17.20}
$$

Because of (17.19), it is sufficient to choose the reference model matrix A_{ref} and the proper stable filter $G(s)$ such that the small gain condition

$$
\gamma_{H_1} < \frac{1}{K_x^{\max}} \tag{17.21}
$$

is satisfied.

Remark 17.1
The upper bound in (17.21) provides guidelines for selecting the reference model matrix A_{ref} and the filter $G(s)$. Basically, these quantities needs to be chosen to minimize the L_1 gain γ_{H_1} of the system H_1 in (17.12).

Based on the above arguments and continuing from the inequality (16.20), one can state and prove the following lemma.

Lemma 17.2
Let the proper stable filter $G(s)$, the Hurwitz matrix A_{ref}, and the Projection Operator bounds $\left(K_x^{\max}, \Lambda^{\max}\right)$ be chosen such that (17.21) holds. Then all trajectories of the closed-loop system (16.6), the reference model (16.10), and the tracking error dynamics are uniformly ultimately bounded. Moreover, the tracking error dynamics are globally asymptotically stable.

Proof: Since the external command r is bounded then x_{ref}^* is UUB. Also, because of (16.20), the tracking error e is UUB. Furthermore, since the assumed inequality (17.21) implies (17.20), the reference model error dynamics (17.11) are input-to-state stable and, consequently e_{ref} is UUB. Hence, as it follows from (17.9), x_{ref} is UUB. Thus, the definition of the tracking error in (16.11), implies that x is UUB. All these facts allow for the application of the Barbalat's Lemma to (16.20), proving UUB of all the signals and asymptotic convergence of the tracking error $e(t)$ to the origin. The proof is complete.

We now summarize the filtered MRAC design and its stability properties.

Theorem 17.1
The filtered MRAC control architecture consists of the filtered feedback signal (16.5),

$$u = G(s)\underbrace{\hat{K}_x^T x}_{u_c} = G(s)u_c$$

the reference model (16.10),

$$\dot{x}_{ref} = A_{ref}\, x_{ref} + B\,\hat{\Lambda}\left(u - u_c\right) + B_c\, r$$

and the Projection Operator based adaptive laws (16.19).

$$\begin{cases} \dot{\hat{K}}_x = -\Gamma_x \operatorname{Proj}\left(\hat{K}_x,\ x e^T P B\right) & , \quad \hat{K}_x(0) = K_{x\,BL} \\ \dot{\hat{\Lambda}} = \operatorname{Proj}\left(\hat{\Lambda},\ B^T P e\left(u - u_c\right)^T\right)\Gamma_\Lambda & , \quad \hat{\Lambda}(0) = I_{m\times m} \end{cases}$$

If the control filter $G(s)$, the reference matrix A_{ref}, and the Projection Operator bounds $\left(K_x^{\max}, \Lambda^{\max}\right)$ are chosen to satisfy the small gain condition in (17.21)

$$\gamma_{H_1} < \frac{1}{K_x^{\max}}$$

where γ_{H_1} is the L_1 gain of the system H_1 in (17.12)

$$e_{ref} = H_1 u_c \Leftrightarrow \dot{e}_{ref} = A_{ref}\, e_{ref} + B\hat{\Lambda}\left(G(s) - I_{m\times m}\right)u_c$$

then:

- The system state x asymptotically tracks the state x_{ref} of the reference model (16.10), when the latter is driven by any bounded reference command $r(t)$.
- The system state vector x and the control input u are UUB.
- The corresponding closed-loop dynamics remain stable with all of its internal signals bounded, uniformly in time.

Example 17.1

Consider open-loop unstable $(A = a > 0)$ scalar LTI system dynamics and suppose that $b = \Lambda = 1$.

$$\dot{x} = a\,x + u$$

Also suppose that

$$A_{ref} = -a_{ref} = a + k_x < 0$$

and that a low-pass filter $G(s)$ is chosen in the form:

$$\frac{u}{u_c} = G(s) = \frac{a_G}{s + a_G}$$

with $a_G > 0$. Then the system H_1 dynamics are

$$e_{ref} = H_1 u_c \Leftrightarrow \dot{e}_{ref} = -a_{ref}\, e_{ref} - \left(\frac{s}{s+a_G}\right) u_c \Leftrightarrow e_{ref} = -\left(\frac{s}{\left(s+a_{ref}\right)\left(s+a_G\right)}\right) u_c$$

Consequently,

$$H_1 = -\frac{s}{\left(s+a_{ref}\right)\left(s+a_G\right)}$$

The system impulse response function $h_1(t)$ can be directly computed via the inverse Laplace transform:

$$\begin{aligned} h_1(t) &= -L^{-1}\left\{\frac{s}{\left(s+a_{ref}\right)\left(s+a_G\right)}\right\} \\ &= \frac{1}{a_G - a_{ref}}\left(L^{-1}\left\{\frac{a_{ref}}{\left(s+a_{ref}\right)}\right\} - L^{-1}\left\{\frac{a_G}{\left(s+a_G\right)}\right\}\right) = \frac{1}{a_G - a_{ref}}\left(a_{ref}\, e^{-a_{ref}\, t} - a_G\, e^{-a_G\, t}\right) \end{aligned}$$

In this case, the L_1 gain of the system H_1 is

$$\gamma_{H_1} = \int_0^\infty \left|h_1(t)\right| dt = \frac{1}{\left|a_G - a_{ref}\right|}\int_0^\infty \left|a_{ref}\, e^{-a_{ref}\, t} - a_G\, e^{-a_G\, t}\right| dt \le \frac{2}{\left|a_G - a_{ref}\right|}$$

Thus, for a given a_{ref}, it is sufficient to choose a_G such that the small gain condition (17.21) is satisfied:

$$\frac{2}{\left|a_G - a_{ref}\right|} < \frac{1}{k_x^{\max}}$$

Consequently, the filter constant a_G must be chosen sufficiently large, so that:

$$\left|a_G - a_{ref}\right| > 2\, k_x^{\max}$$

or, equivalently

$$a_G > 2\, k_x^{\max} + a_{ref} = a + 3\, k_x^{\max}$$

Suppose that the plant contains a 1st order actuator model, whose transfer function is:

$$G_{act}(s) = \frac{a_{act}}{s + a_{act}}$$

where $a_{act} > 0$ is the inverse of the actuator time constant. In this case, the H_1 dynamics become:

$$e_{ref} = H_1 u_c \Leftrightarrow e_{ref} = -\left(\frac{s\, a_{act}}{(s + a_{ref})(s + a_{act})(s + a_G)} \right) u_c$$

The impulse response function of H_1 is:

$$h_1 = a_{act} \left(\frac{a_{ref}\, e^{-a_{ref} t}}{(a_{act} - a_{ref})(a_G - a_{ref})} + \frac{a_{act}\, e^{-a_{act} t}}{(a_G - a_{act})(a_{ref} - a_{act})} + \frac{a_G\, e^{-a_G t}}{(a_{ref} - a_G)(a_{act} - a_G)} \right)$$

Consider the case when

$$a_{ref} < a_G < a_{act}$$

Then the impulse response function becomes:

$$h_1 = a_{act} \left(\frac{a_{ref}\, e^{-a_{ref} t}}{|a_{act} - a_{ref}||a_G - a_{ref}|} + \frac{a_{act}\, e^{-a_{act} t}}{|a_G - a_{act}||a_{ref} - a_{act}|} - \frac{a_G\, e^{-a_G t}}{|a_{ref} - a_G||a_{act} - a_G|} \right)$$

Now, the system L_1 gain can be upper bounded:

$$\begin{aligned}
\gamma_{H_1} &= \int_0^\infty |h(t)|\, dt \\
&\le a_{act} \left(\frac{1}{|a_{act} - a_{ref}||a_G - a_{ref}|} + \frac{1}{|a_G - a_{act}||a_{ref} - a_{act}|} + \frac{1}{|a_{ref} - a_G||a_{act} - a_G|} \right) \\
&= \frac{2\, a_{act}}{(a_G - a_{ref})(a_{act} - a_G)}
\end{aligned}$$

and, thus

$$\gamma_{H_1} \le \frac{2\, a_{act}}{(a_G - a_{ref})(a_{act} - a_G)}$$

Since our goal is to minimize the L_1 gain of H_1, we choose a_G such that the upper bound above is minimized. The latter is equivalent to maximizing the following expression:

$$\left(a_G - a_{ref}\right)\left(a_{act} - a_G\right) \to \min_{a_G}$$

It is easy to see that the optimal solution is:

$$a_G = \frac{a_{ref} + a_{act}}{2}$$

In this case, the L_1 gain upper bound becomes:

$$\gamma_{H_1} \le \frac{8\,a_{act}}{\left(a_{act} - a_{ref}\right)^2}$$

At the same time, the small gain condition requires that

$$\frac{8\,a_{act}}{\left(a_{act} - a_{ref}\right)^2} < \frac{1}{k_x^{\max}}$$

When given the actuator dynamics, the above inequality constraints the selection of the reference model dynamics which is achievable under the filtered MRAC design.