Lecture 8

17. Stability Analysis

For an \( n \) - dimensional vector \( x \in \mathbb{R}^n \), the \( L_2 \) and the \( L_\infty \) vector norms are defined as:

\[
\|x\| = \sqrt{x^T x} \\
\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|
\]  

(17.1)

It is easy to show that these two norms satisfy the following relation:

\[
\|x\|_\infty \leq \|x\| \leq \sqrt{n} \|x\|_{\infty}
\]  

(17.2)

If a vector \( x \) is time-dependent, then its truncated \( L_\infty \) functional norms is defined as:

\[
\|x(t)\|_{\infty} = \max_{t_0 \leq \tau \leq t} \|x(\tau)\|
\]  

(17.3)

Note that in (17.3), the \( L_2 \) vector norm is used instead of the \( L_\infty \) vector norm. For a matrix \( A \), vector induced \( L_2 \) and \( L_\infty \) matrix norms will be denoted by \( \|A\| \) and \( \|A\|_{\infty} \), correspondingly. Suppose that

\[
K^\max_x = \max_{t \geq 0} \|\hat{K}_x(t)\| \\
\Lambda^\max = \max_{t \geq 0} \|\hat{\Lambda}(t)\|
\]  

(17.4)

are the parameter bounds for the Projection Operator used in the adaptive laws (16.19).

First, we state and prove the following fact about input-to-state stability of a linear time-invariant (LTI) system.

**Lemma 17.1**

There exist strictly positive constants \( (k_A, \lambda_A) \), such that starting at any initial condition \( x(0) = x_0 \), the state vector \( x(t) \in \mathbb{R}^n \) of the LTI system

\[
\dot{x} = Ax + Bu
\]  

(17.5)

with a Hurwitz matrix \( A \) and an external input \( u(t) \in \mathbb{R}^m \), satisfies
\[ \|x(t)\|_{\infty} \leq k_A \left( \|x_0\| + \frac{\|B\|}{\lambda_A} \|u(t)\|_{\infty} \right) \] (17.6)

**Proof:** Since \( A \) is Hurwitz then there must exist strictly positive constants \((k_A, \lambda_A)\) such that \( \|e^{A t}\| \leq k_A e^{-\lambda_A t} \). In this case, truncated \( L_{\infty} \) norm upper bound for the solution of (17.5) can be found.

\[
\|x(t)\|_{\infty} = \max_{0 \leq t \leq T} \|e^{A t} x_0 + \int_0^t e^{(t-\tau)A} B u(\tau) d\tau\|
\]
\[
\leq \max_{0 \leq t \leq T} \left( k_A e^{-\lambda_A t} \|x_0\| + k_A \|B\| \int_0^t e^{-\lambda_A (t-\tau)} \|u(\tau)\| d\tau \right)
\]
\[
\leq k_A \left( \|x_0\| + \|B\| \int_0^t e^{-\lambda_A (t-\tau)} \|u(\tau)\| d\tau \right) \leq k_A \left( \|x_0\| + \|B\| \int_0^T e^{-\lambda_A (t-\tau)} d\tau \max_{0 \leq \tau \leq T} \|u(\tau)\| \right)
\]
\[
\leq k_A \left( \|x_0\| + \|B\| \lambda_A \|u(\tau)\|_{\infty} \right)
\]

This completes the proof.

Next, we find sufficient conditions to guarantee uniform ultimate boundedness (UUB) of all the signals in the system (15.1), which is controlled by a filtered MRAC controller, in the form of (16.5), (16.10) and (16.19). Towards that end, introduce the so-called ideal reference model

\[ \dot{x}_{\text{ref}}^* = A_{\text{ref}} x_{\text{ref}}^* + B_c r(t) \] (17.8)

and let

\[ e_{\text{ref}} = x_{\text{ref}} - x_{\text{ref}}^* \] (17.9)

represent the error between the actual reference model (16.10) and its ideal target (17.8). Then the reference model error dynamics can be computed as:

\[ \dot{e}_{\text{ref}} = A_{\text{ref}} e_{\text{ref}} + B \hat{\Lambda} \left( G(s) - I_{\text{sum}} \right) \hat{K}_x^T x_{\hat{\theta}} \] (17.10)

Using (16.11) and (17.9), gives

\[ \dot{e}_{\text{ref}} = A_{\text{ref}} e_{\text{ref}} + B \hat{\Lambda} \left( G(s) - I_{\text{sum}} \right) \hat{K}_x^T \left( e_{\text{ref}} + x_{\text{ref}}^* + e \right) \] (17.11)

where \( \hat{\Lambda}, \hat{K}_x, x_{\text{ref}}^* \) and \( e \) are UUB. We will use Small Gain Theorem type arguments to find sufficient conditions for input-to-state stability of (17.11). This system can be
represented by the two feedback-connected systems $H_1$ and $H_2$, as shown in the block-diagram:

![Block-Diagram](image)

Figure 1: Reference Model Error Dynamics

According to the figure, these two systems are defined as

$$ e_{ref} = H_1 u_c \Leftrightarrow \dot{e}_{ref} = A_{ref} e_{ref} + B \hat{\Lambda} (G(s) - I_{nxm}) u_c \quad (17.12) $$

and

$$ u_c = H_2 x \Leftrightarrow u_c = \hat{K}_x^T (e_{ref} + x_{ref}^* + e) \quad (17.13) $$

Note that $H_1$ is driven by the commanded control input

$$ u_c = \hat{K}_x^T (e_{ref} + x_{ref}^* + e) \quad (17.14) $$

while $H_2$ is driven by the system state vector

$$ x = e_{ref} + (x_{ref}^* + e) \quad (17.15) $$

Let $\gamma_{H_1}$ denote the $L_1$ gain of the system $H_1$. One can easily show that the gain value is finite. In fact, applying Lemma 17.1 to (17.12) gives
\[ \| e_{\text{ref}}(t) \|_{L_\infty} \leq k_{\text{ref}} \left( \| e_{\text{ref}}(0) \| + \frac{\| B \| \Lambda_{\max}}{\lambda_{\text{ref}}} \left( \| G(s) - I_{m \times m} \|_{L_\infty} u_c \right) \right) \]

\[ \leq k_{\text{ref}} \left( \| e_{\text{ref}}(0) \| + \frac{\| B \| \Lambda_{\max}}{\lambda_{\text{ref}}} \left( \| G(s) u_c \|_{L_\infty} + \| u_c \|_{L_\infty} \right) \right) \]

\[ \leq k_{\text{ref}} \| e_{\text{ref}}(0) \| + \frac{k_{\text{ref}}}{\lambda_{\text{ref}}} \| B \| \Lambda_{\max} (\gamma_G + 1) \| u_c \|_{L_\infty} \]  

(17.16)

where \( \gamma_G \) is the \( L_1 \) gain of the proper and stable filter \( G(s) \). The inequality in (17.16) gives finite upper bound for the \( L_1 \) gain \( \gamma_{H_1} \) of the system \( H_1 \):

\[ \gamma_{H_1} \leq \frac{k_{\text{ref}}}{\lambda_{\text{ref}}} |B| \Lambda_{\max} (\gamma_G + 1) \]  

(17.17)

Furthermore, the relations

\[ \| e_c \|_{L_\infty} = \left\| \tilde{K}_x^T (e_{\text{ref}} + x_{\text{ref}}^* + e) \right\|_{L_\infty} \leq K_x \left( \| e_{\text{ref}}(t) \|_{L_\infty} + \left\| \left( x_{\text{ref}}(t) \right)_{\text{max}} \right\|_{L_\infty} + \left\| \left( e(t) \right)_{\text{max}} \right\|_{L_\infty} \right) \]

\[ \leq K_x \left( \| e_{\text{ref}}(t) \|_{L_\infty} + K_x \left( x_{\text{ref}}(\max) + e_{\text{max}} \right) \right) \]  

(17.18)

imply that the \( L_1 \) gain \( \gamma_{H_2} \) of the system \( H_2 \) is also finite:

\[ \gamma_{H_2} \leq K_x \]  

(17.19)

Finally, using Small Gain Theorem allows to claim input-to-state stability of the system (17.11), if

\[ \gamma_{H_1} \gamma_{H_2} < 1 \]  

(17.20)

Because of (17.19), it is sufficient to choose the reference model matrix \( A_{\text{ref}} \) and the proper stable filter \( G(s) \) such that the small gain condition

\[ \gamma_{H_1} < \frac{1}{K_x \max} \]  

(17.21)

is satisfied.
Remark 17.1
The upper bound in (17.21) provides guidelines for selecting the reference model matrix \( A_{\text{ref}} \) and the filter \( G(s) \). Basically, these quantities need to be chosen to minimize the \( L_1 \) gain \( \gamma_{H_1} \) of the system \( H_1 \) in (17.12).

Based on the above arguments and continuing from the inequality (16.20), one can state and prove the following lemma.

Lemma 17.2
Let the proper stable filter \( G(s) \), the Hurwitz matrix \( A_{\text{ref}} \), and the Projection Operator bounds \( (K_x^{\max}, \Lambda^{\max}) \) be chosen such that (17.21) holds. Then all trajectories of the closed-loop system (16.6), the reference model (16.10), and the tracking error dynamics are uniformly ultimately bounded. Moreover, the tracking error dynamics are globally asymptotically stable.

Proof: Since the external command \( r \) is bounded then \( x^*_{\text{ref}} \) is UUB. Also, because of (16.20), the tracking error \( e \) is UUB. Furthermore, since the assumed inequality (17.21) implies (17.20), the reference model error dynamics (17.11) are input-to-state stable and, consequently \( e_{\text{ref}} \) is UUB. Hence, as it follows from (17.9), \( x_{\text{ref}} \) is UUB. Thus, the definition of the tracking error in (16.11), implies that \( x \) is UUB. All these facts allow for the application of the Barbalat’s Lemma to (16.20), proving UUB of all the signals and asymptotic convergence of the tracking error \( e(t) \) to the origin. The proof is complete.

We now summarize the filtered MRAC design and its stability properties.

Theorem 17.1
The filtered MRAC control architecture consists of the filtered feedback signal (16.5),

\[
    u = G(s)\frac{K^T}{u_c} x = G(s)u_c 
\]

the reference model (16.10),

\[
    \dot{x}_{\text{ref}} = A_{\text{ref}} x_{\text{ref}} + B\hat{A}(u - u_c) + B_c r 
\]

and the Projection Operator based adaptive laws (16.19).
If the control filter $G(s)$, the reference matrix $A_{\text{ref}}$, and the Projection Operator bounds $(K_{x}^{\max}, \Lambda^{\max})$ are chosen to satisfy the small gain condition in (17.21)

$$\gamma_{H_{1}} < \frac{1}{K_{x}^{\max}}$$

where $\gamma_{H_{1}}$ is the $L_1$ gain of the system $H_{1}$ in (17.12)

$$e_{\text{ref}} = H_{1} u_{c} \Leftrightarrow \dot{e}_{\text{ref}} = A_{\text{ref}} e_{\text{ref}} + B \hat{\Lambda} (G(s) - I_{m\times m}) u_{c}$$

then:

- The system state $x$ asymptotically tracks the state $x_{\text{ref}}$ of the reference model (16.10), when the latter is driven by any bounded reference command $r(t)$.

- The system state vector $x$ and the control input $u$ are UUB.

- The corresponding closed-loop dynamics remain stable with all of its internal signals bounded, uniformly in time.

**Example 17.1**

Consider open-loop unstable ($A = a > 0$) scalar LTI system dynamics and suppose that $b = \Lambda = 1$.

$$\dot{x} = ax + u$$

Also suppose that

$$A_{\text{ref}} = -a_{\text{ref}} = a + k_{x} < 0$$

and that a low-pass filter $G(s)$ is chosen in the form:

$$\frac{u}{u_{c}} = G(s) = \frac{a_{G}}{s + a_{G}}$$
with \( a_G > 0 \). Then the system \( H_1 \) dynamics are

\[
e_{\text{ref}} = H_1 u_e \Leftrightarrow \dot{e}_{\text{ref}} = -a_{\text{ref}} e_{\text{ref}} - \left( \frac{s}{s + a_G} \right) u_e \Leftrightarrow e_{\text{ref}} = -\left( \frac{s}{(s + a_{\text{ref}})(s + a_G)} \right) u_e
\]

Consequently,

\[
H_1 = -\frac{s}{(s + a_{\text{ref}})(s + a_G)}
\]

The system impulse response function \( h_1(t) \) can be directly computed via the inverse Laplace transform:

\[
h_1(t) = -L^{-1}\left\{ \frac{s}{(s + a_{\text{ref}})(s + a_G)} \right\} = -\frac{1}{a_G - a_{\text{ref}}} \left( L^{-1}\left\{ \frac{a_{\text{ref}}}{s + a_{\text{ref}}} \right\} - L^{-1}\left\{ \frac{a_G}{s + a_G} \right\} \right) = \frac{1}{a_G - a_{\text{ref}}} \left( a_{\text{ref}} e^{-a_{\text{ref}} t} - a_G e^{-a_G t} \right)
\]

In this case, the \( L_1 \) gain of the system \( H_1 \) is

\[
\gamma_{H_1} = \left\| h_1(t) \right\| dt = \frac{1}{a_G - a_{\text{ref}}} \left[ a_{\text{ref}} e^{-a_{\text{ref}} t} - a_G e^{-a_G t} \right] dt \leq \frac{2}{a_G - a_{\text{ref}}}
\]

Thus, for a given \( a_{\text{ref}} \), it is sufficient to choose \( a_G \) such that the small gain condition (17.21) is satisfied:

\[
\frac{2}{a_G - a_{\text{ref}}} \leq \frac{1}{k_x^{\max}}
\]

Consequently, the filter constant \( a_G \) must be chosen sufficiently large, so that:

\[
|a_G - a_{\text{ref}}| > 2k_x^{\max}
\]

or, equivalently

\[
a_G > 2k_x^{\max} + a_{\text{ref}} = a + 3k_x^{\max}
\]
Suppose that the plant contains a 1st order actuator model, whose transfer function is:

\[ G_{\text{act}}(s) = \frac{a_{\text{act}}}{s + a_{\text{act}}} \]

where \( a_{\text{act}} > 0 \) is the inverse of the actuator time constant. In this case, the \( H_1 \) dynamics become:

\[
e_{\text{ref}} = H_1 u_c \Leftrightarrow e_{\text{ref}} = -\left( \frac{s a_{\text{act}}}{(s + a_{\text{ref}})(s + a_{\text{act}})(s + a_G)} \right) u_c
\]

The impulse response function of \( H_1 \) is:

\[
h_1 = a_{\text{act}} \left( \frac{a_{\text{ref}} e^{-a_{\text{ref}} t}}{(a_{\text{act}} - a_{\text{ref}})(a_G - a_{\text{ref}})} + \frac{a_{\text{act}} e^{-a_{\text{act}} t}}{(a_G - a_{\text{act}})(a_{\text{ref}} - a_{\text{act}})} + \frac{a_G e^{-a_G t}}{(a_{\text{ref}} - a_G)(a_{\text{act}} - a_G)} \right)
\]

Consider the case when

\[ a_{\text{ref}} < a_G < a_{\text{act}} \]

Then the impulse response function becomes:

\[
h_1 = a_{\text{act}} \left( \frac{a_{\text{ref}} e^{-a_{\text{ref}} t}}{(a_{\text{act}} - a_{\text{ref}})(a_G - a_{\text{ref}})} + \frac{a_{\text{act}} e^{-a_{\text{act}} t}}{(a_G - a_{\text{act}})(a_{\text{ref}} - a_{\text{act}})} - \frac{a_G e^{-a_G t}}{(a_{\text{ref}} - a_G)(a_{\text{act}} - a_G)} \right)
\]

Now, the system \( L_1 \) gain can be upper bounded:

\[
\gamma_{H_1} = \int_0^\infty |h(t)| dt 
\leq a_{\text{act}} \left( \frac{1}{|a_{\text{act}} - a_{\text{ref}}||a_G - a_{\text{ref}}|} + \frac{1}{|a_G - a_{\text{act}}||a_{\text{ref}} - a_{\text{act}}|} + \frac{1}{|a_{\text{ref}} - a_G||a_{\text{act}} - a_G|} \right) 
= \frac{2 a_{\text{act}}}{(a_G - a_{\text{ref}})(a_{\text{act}} - a_G)}
\]

and, thus

\[
\gamma_{H_1} \leq \frac{2 a_{\text{act}}}{(a_G - a_{\text{ref}})(a_{\text{act}} - a_G)}
\]
Since our goal is to minimize the $L_1$ gain of $H_1$, we choose $a_G$ such that the upper bound above is minimized. The latter is equivalent to maximizing the following expression:

$$\left( a_G - a_{\text{ref}} \right) \left( a_{\text{act}} - a_G \right) \rightarrow \min_{a_G}$$

It is easy to see that the optimal solution is:

$$a_G = \frac{a_{\text{ref}} + a_{\text{act}}}{2}$$

In this case, the $L_1$ gain upper bound becomes:

$$\gamma_{H_1} \leq \frac{8 a_{\text{act}}}{\left( a_{\text{act}} - a_{\text{ref}} \right)^2}$$

At the same time, the small gain condition requires that

$$\frac{8 a_{\text{act}}}{\left( a_{\text{act}} - a_{\text{ref}} \right)^2} < \frac{1}{k_s^\text{max}}$$

When given the actuator dynamics, the above inequality constraints the selection of the reference model dynamics which is achievable under the filtered MRAC design.