Lecture 7

Note that the knowledge of the true gains will not be required. From (14.10) - (14.11), it immediately follows that:

$$\begin{cases} \hat{A} - A_{ref} = B_1 \Lambda \underbrace{\left(K_x + \hat{K}_x - K_x^* \right)}_{\Delta K_x}^T = B_1 \Lambda \Delta K_x^T \\ \hat{B} - B_{ref} = B_1 \Lambda \underbrace{\left(K_r + \hat{K}_r - K_r^* \right)}_{\Delta K_r}^T = B_1 \Lambda \Delta K_r^T \end{cases} \tag{14.12}$$

Using (14.6), (14.8), (14.10), and(14.12), the tracking error dynamics can be written as:

$$\dot{e} = A_{ref}\, e + B_1 \Lambda \left(\Delta K_x^T\, x + \Delta K_r^T\, r - \Delta f\left(x_p \right) \right) \tag{14.13}$$

The plant uncertainty $f\left(x_p\right)$ is approximated using an RBF NN, i.e., linear combinations of N suitably chosen radial basis functions $\left\{ \varphi_j\left(x_p\right) \right\}_{j=1}^{N}$.

$$\hat{f}\left(x_p\right) = \begin{pmatrix} \sum_{j=1}^{N} \hat{\theta}_{j1}\, \varphi_j\left(x_p\right) \\ \dots \\ \sum_{j=1}^{N} \hat{\theta}_{jm}\, \varphi_j\left(x_p\right) \end{pmatrix} = \hat{\Theta}^T\, \Phi\left(x_p\right) \tag{14.14}$$

RBF NN Universal Approximation Theorem: Given approximation tolerance $\varepsilon_0^* > 0$, and a compact set $X \subset R^n$, there exist an integer N and a "true" constant matrix $\Theta^* \in R^{N \times m}$ such that for all $x_p \in X \subset R^n$:

$$f\left(x_p\right) = \left(\Theta^*\right)^T \Phi\left(x_p\right) + \varepsilon_f\left(x_p\right) \tag{14.15}$$

and

$$\left\| \varepsilon_f\left(x_p\right) \right\| \le \varepsilon_f^{\max} \tag{14.16}$$

Using (15), we get

$$\Delta f\left(x_p\right) = \hat{f}\left(x_p\right) - f\left(x_p\right) = \underbrace{\left(\hat{\Theta} - \Theta\right)}_{\Delta\Theta}^T \Phi\left(x_p\right) - \varepsilon_f\left(x_p\right) \tag{14.17}$$

In (14.17), $\Delta\Theta = \hat{\Theta} - \Theta$ represents the parameter estimation errors, while $\varepsilon_f\left(x_p\right)$ is the function approximation error. Substituting (14.17) into the tracking error dynamics equation (14.13), further yields:

$$\dot{e} = A_{ref}\, e + B_1\, \Lambda\left(\Delta K_x^T\, x + \Delta K_r^T\, r - \Delta\Theta^T\, \Phi\left(x_p\right) + \varepsilon_f\left(x_p\right)\right) \tag{14.18}$$

Since the tracking error dynamics (14.18) is similar to the tracking error dynamics in (11.9), the adaptive laws can be written in the form of (11.17):

$$\begin{cases} \dot{\hat{K}}_x = \Gamma_x\, \mathrm{Proj}\left(\hat{K}_x,\, -x\, e^T P\, B_1\right), \quad \hat{K}_x\left(0\right) = 0 \\ \dot{\hat{K}}_r = \Gamma_r\, \mathrm{Proj}\left(\hat{K}_r,\, -r\left(t\right) e^T P\, B_1\right), \quad \hat{K}_r\left(0\right) = 0 \\ \dot{\hat{\Theta}} = \Gamma_\Theta\, \mathrm{Proj}\left(\hat{\Theta},\, \Phi\left(x\right) e^T P\, B_1\right), \quad \dot{\hat{\Theta}}\left(0\right) = 0 \end{cases} \tag{14.19}$$

Repeating the UUB analysis from Section 13, it can be proven that the adaptive laws provide bounded tracking with all the signals bounded.

<u>In summary</u>, total control u is composed of the baseline component u_{bl}, which is augmented by the <u>*incremental direct adaptive control*</u> u_{ad}, that is:

$$u = u^{bl} + u^{ad}$$

where:

$$u_{bl} = K_x^T\, x + K_r^T\, r$$

and

$$u_{ad} = \hat{K}_x^T\, x + \hat{K}_r^T\, r\left(t\right) - \hat{\Theta}^T\, \Phi\left(x_p\right)$$

with the Projection Operator based <u>*incremental adaptive laws*</u> as in (14.19). At the same time, the *Projection Operator* is defined column-wise, for any two vectors θ and y:

$$\mathrm{Proj}\left(\theta,\, y\right) = \begin{cases} y - \dfrac{\nabla f\left(\theta\right)\left(\nabla f\left(\theta\right)\right)^T}{\left\|\nabla f\left(\theta\right)\right\|^2}\, y\, f\left(\theta\right), & \text{if } \left[\, f\left(\theta\right) > 0 \;\wedge\; y^T \nabla f\left(\theta\right) > 0 \,\right] \\ y, & \text{if not} \end{cases}$$

with convex function $f\left(\bullet\right)$ and its gradient

$$\begin{cases} f(\theta) = \dfrac{\theta^T\theta - \theta_{\max}^2}{\varepsilon_\theta\,\theta_{\max}^2} \\ \nabla f(\theta) = \dfrac{2}{\varepsilon_\theta\,\theta_{\max}^2}\theta \end{cases} \tag{14.20}$$

where $\theta_{\max}$ represents the allowable upper bound imposed on the adaptive parameter vector θ, and ε_θ denotes the projection tolerance.

The MRAC augmentation concept is sketched in Figure 14.1, for an aircraft related autopilot design problem.

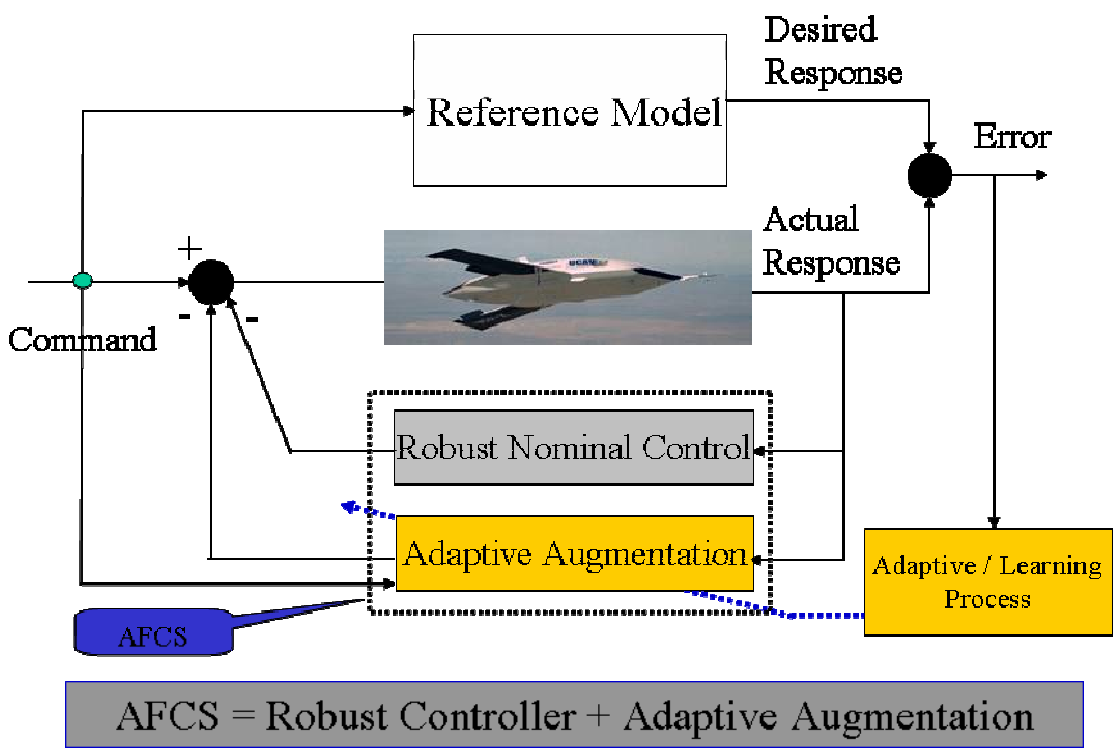

Figure 14.1: MRAC Augmentation of a Baseline / Nominal Controller

References

1. J.J. Slotine, W. Li, Applied Nonlinear Control, Prentice Hall, 1995.
2. H.K. Khalil, Nonlinear Systems, 3rd Edition, Prentice Hall, New Jersey, 2000.
3. S. Haykin, Neural Networks: A Comprehensive Foundation, 2nd Edition, Prentice Hall, New Jersey, 1991.

15. Filtered MRAC Design

Consider the linear dynamical system

$$\begin{aligned} \dot{x} &= A\,x + B\,\Lambda\,u + B_c\,r(t) \\ y &= C\,x \end{aligned} \tag{15.1}$$

where $x \in R^n$ is the n – dimensional system state, $u \in R^m$ is the m – dimensional control input, y is the m – dimensional system output, r is the m – dimensional external command, and matrices A, B, B_c, C, are of the corresponding dimensions. It is assumed that (A, B) is controllable, (C, A) is observable, $A \in R^{n\times n}$ is *unknown*, and $\Lambda \in R^{m\times m}$ is an unknown diagonal positive-definite matrix.

Motivation for considering systems in the form of (15.1) comes from solving *servomechanism* problems where the main goal is to track constant external signals without steady state errors. In order to do that, one would augment the system dynamics with an integrated output tracking error and thus arrive at (15.1).

To reiterate, the *control goal* of interest is *bounded tracking* in the presence of uncertainties, that is one needs to find u such that the output y asymptotically tracks its external command r, in the presence of the system uncertain matrices A and Λ, while keeping all the signals in (15.1) bounded.

16. Control Design

Consider a class of Hurwitz matrices satisfying the system *matching conditions*:

$$\mathrm{A}_{ref} = \left\{ A_{ref} \in R^{n\times n} : \ \forall \Lambda = \Lambda^T > 0,\ \exists K_x \in R^{n\times m},\ \rho\left(A_{ref}\right) = \rho\left(A + B\,\Lambda\,K_x^T\right) \subset \mathbb{C}^- \right\} \tag{16.1}$$

Choose $A_{ref} \in \mathrm{A}_{ref}$. Then for a constant unknown positive-definite diagonal matrix Λ, there must exist a constant, possibly unknown, gain matrix K_x such that

$$A_{ref} = A + B\,\Lambda\,K_x^T \tag{16.2}$$

is Hurwitz. This allows to rewrite the open-loop dynamics (15.1) as

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(u - K_x^T\,x\right) + B_c\,r \tag{16.3}$$

Assume that B_c is chosen such that

$$-C\,A_{ref}^{-1}\,B_c = I_{m\times m} \tag{16.4}$$

Let $G(s) \in R^{m\times m}$ be a transfer function (a filter), whose selection will be specified later. Control input is formed as:

$$u = G(s)\underbrace{\hat{K}_x^T\,x}_{u_c} = G(s)\,u_c \tag{16.5}$$

where $\hat{K}_x$ is the adaptive gain matrix and u_c denotes the commanded control feedback signal. Substituting (16.5) into (16.3) yields the closed-loop dynamics:

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(G(s)\,\hat{K}_x^T - K_x^T\right)x + B_c\,r \tag{16.6}$$

Let

$$\begin{cases} \Delta K_x = \hat{K}_x - K_x \\ \Delta\Lambda = \hat{\Lambda} - \Lambda \end{cases} \tag{16.7}$$

denote parameter estimation errors, where $\hat{\Lambda} \in R^{m\times m}$ is a diagonal adaptive gain matrix to be specified later.

In terms of (16.7), the dynamics (16.6) can be expressed as:

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(\left(G(s) - I_{m\times m}\right)\underbrace{\hat{K}_x^T\,x}_{u_c} + \Delta K_x^T\,x\right) + B_c\,r \tag{16.8}$$

or, equivalently

$$\dot{x} = A_{ref}\,x + B\,\Lambda\left(u - u_c + \Delta K_x^T\,x\right) + B_c\,r \tag{16.9}$$

Based on (16.9), the *reference model* is chosen as:

$$\dot{x}_{ref} = A_{ref}\,x_{ref} + B\,\hat{\Lambda}\left(u - u_c\right) + B_c\,r \tag{16.10}$$

Note that the dynamics (16.10) depends on the system state x. Define system tracking error

$$e = x - x_{ref} \tag{16.11}$$

Its dynamics is calculated by subtracting (16.10) from (16.8).

$$\dot{e} = A_{ref}\, e + B\left(\Lambda\, \Delta K_x^T\, x - \Delta\Lambda\left(u - u_c\right)\right) \tag{16.12}$$

Consider Lyapunov function candidate:

$$V\left(e, \Delta K_x, \Delta\Lambda\right) = e^T\, P\, e + \mathrm{trace}\left(\Delta K_x^T\, \Gamma_x^{-1}\, \Delta K_x\, \Lambda\right) + \mathrm{trace}\left(\Delta\Lambda\, \Gamma_\Lambda^{-1}\, \Delta\Lambda^T\right) \tag{16.13}$$

where $\Gamma_x = \Gamma_x^T > 0$ and $\Gamma_\Lambda = \Gamma_\Lambda^T > 0$ are the rates of adaptations, while $P = P^T > 0$ is the unique solution of the algebraic Lyapunov equation

$$A_{ref}^T\, P + P\, A_{ref} = -Q \tag{16.14}$$

with some $Q = Q^T > 0$. Differentiating (16.13) along the trajectories of (16.12), yields

$$\begin{aligned} \dot{V} &= -e^T\, Q\, e + 2\, e^T\, P\, B\left(\Lambda\, \Delta K_x^T - \Delta\Lambda\left(u - u_c\right)\right) x \\ &+ \mathrm{trace}\left(\Delta K_x^T\, \Gamma_x^{-1}\, \dot{\hat{K}}_x\, \Lambda\right) + \mathrm{trace}\left(\Delta\Lambda^T\, \Gamma_\Lambda^{-1}\, \dot{\hat{\Lambda}}^T\right) \end{aligned} \tag{16.15}$$

The trace identity

$$a^T\, b = trace\left(b\, a^T\right) \tag{16.16}$$

is valid for any two column vectors of the same dimension. Using (16.16), one gets

$$\begin{aligned} \dot{V} &= -e^T\, Q\, e + 2\,\mathrm{trace}\left(\Delta K_x^T\, x\, e^T\, P\, B\, \Lambda\right) + 2\,\mathrm{trace}\left(\Delta K_x^T\, \Gamma_x^{-1}\, \dot{\hat{K}}_x\, \Lambda\right) \\ &- 2\,\mathrm{trace}\left(\Delta\Lambda\left(u - u_c\right) x\, e^T\, P\, B\right) + 2\,\mathrm{trace}\left(\Delta\Lambda^T\, \Gamma_\Lambda^{-1}\, \dot{\hat{\Lambda}}^T\right) \end{aligned} \tag{16.17}$$

Collecting similar terms, yields

$$\begin{aligned} \dot{V} &= -e^T\, Q\, e + 2\,\mathrm{trace}\left(\Delta K_x^T\left(x\, e^T\, P\, B + \Gamma_x^{-1}\, \dot{\hat{K}}_x\right)\Lambda\right) \\ &- 2\,\mathrm{trace}\left(\Delta\Lambda\left(\left(u - u_c\right) e^T\, P\, B\right) - \Gamma_\Lambda^{-1}\, \dot{\hat{\Lambda}}^T\right) \end{aligned} \tag{16.18}$$

Consequently, choosing the following adaptive laws

$$\begin{cases} \dot{\hat{K}}_x = -\Gamma_x\, \mathrm{Proj}\left(\hat{K}_x,\ x\, e^T\, P\, B\right) \\ \dot{\hat{\Lambda}} = \mathrm{Proj}\left(\hat{\Lambda},\ B^T P\, e\left(u - u_c\right)^T\right)\Gamma_\Lambda \end{cases} \tag{16.19}$$

gives

$$\dot{V}\left(e, \Delta K_x, \Delta\Lambda\right) = -e^T Q e \leq 0 \tag{16.20}$$

which proves: a) stability of (16.12)-(16.19), b) uniform boundedness of e, ΔK_x, $\Delta\Lambda$, and, consequently c) uniform boundedness of the adaptive parameters $\hat{K}_x$ and $\hat{\Lambda}$.

However, nothing can be said about boundedness of *all* the signals in the closed-loop system (16.8). Also, convergence of the tracking error $e(t)$ can not be asserted. The incomplete stability proof is attributed to the dependence of the reference model (16.10) on the system state x. In order to complete the proof, one needs to establish boundedness of either the reference model state x_{ref} or of the original system state x. This fact would subsequently allow for the application of the Barbalat's Lemma, proving closed-loop system stability, and solving the asymptotic bounded tracking problem.

Remark 16.1
Due to the form of the adaptive laws (16.19), the estimated matrix $\hat{\Lambda}(t)$ is not guaranteed to be diagonal.

Remark 16.2
Although initial values for the adaptive parameters in (16.19) can be chosen arbitrarily, the following selection is recommended:

$$\begin{aligned} \hat{K}_x(0) &= K_{x\,BL} \\ \hat{\Lambda}(0) &= I_{m\times m} \end{aligned} \tag{16.21}$$

where $K_{x\,BL} \in R^{n\times m}$ denotes a baseline feedback gain matrix.