Lecture 4

MRAC Design of 1st Order Systems

Suppose that a plant contains unknown constant parameters, without any information about their bounds. The plant dynamics is

\[ \dot{x} = ax + b(u + f(x)) \]  

(7.4)

where \( x \) is the state, \( u \) is the control input, \( a \) and \( b \) are unknown constants. It is assumed that the sign of \( b \) is known, while the unknown and possibly nonlinear function \( f(x) \) is linearly parameterized in terms of \( N \) unknown constant parameters \( \theta_i \) and known bounded basis functions \( \varphi_i \).

\[ f(x) = \sum_{i=1}^{N} \theta_i \varphi_i(x) = \theta^T \Phi(x) \]  

(7.5)

In (7.5), \( \Phi(x) = (\varphi_1(x), \ldots, \varphi_N(x))^T \in \mathbb{R}^N \) denotes the known regressor vector. It is assumed that the regressor components \( \varphi_i(x) \) are piece-wise continuous functions of the system state \( x \).

A reference model is described by the 1st order differential equation

\[ \dot{x}_m = a_m x_m + b_m r(t) \]  

(7.6)

where \( a_m < 0 \) and \( b_m \) are the desired constants and \( r(t) \) is the reference input.

The task is to design a control law \( u(t) \) such that all the signals in the system remain bounded, while the tracking error \( e(t) = x(t) - x_m(t) \) tends to zero asymptotically, as \( t \to \infty \). Notice, that the tracking task must be accomplished in the presence of \( (N+2) \) unknown constant parameters: \( \{a, b, \theta, \ldots, \theta_N\} \).

First, we define an ideal control solution, as if the unknown parameters were known. The ideal control is formed using feedback / feedforward architecture

\[ u_{\text{ideal}} = k_x x + k_r r - \theta^T \Phi(x) \]  

(7.7)

Substituting (7.7) into (7.4), the closed-loop dynamics can be written.
\[
\dot{x} = (a + bk_x)x + bk_r r(t)
\]  

(7.8)

Comparing (7.8) with the desired reference model dynamics (7.6), it immediately follows that ideal gains \( k_x \) and \( k_r \) must satisfy the following matching conditions:

\[
\begin{align*}
    a + bk_x &= a_m \\
    bk_r &= b_m
\end{align*}
\]  

(7.9)

Since in (7.9) there are 2 equations and two unknowns, it becomes clear that the ideal solution (which is not known!) always exists.

Based on (7.7), tracking control solution is formed.

\[
u = \hat{k}_x x + \hat{k}_r r - \hat{\theta}^T \Phi(x)
\]  

(7.10)

where the feedback gain \( \hat{k}_x \), the feedforward gain \( \hat{k}_r \), and the estimated vector of parameters \( \hat{\theta} \) will be found to achieve the desired tracking.

Towards this end, substitute (7.10) into the system dynamics (7.4). Then the closed-loop system becomes

\[
\dot{x} = (a + b\hat{k}_x)x + b\left(\hat{k}_r r - (\hat{\theta} - \theta)^T \Phi(x)\right)
\]  

(7.11)

Using matching conditions (7.9) yields

\[
\begin{align*}
    \dot{x} &= a_m x + bk_r r + b\left(\hat{k}_x - k_x\right)x + b\left(\hat{k}_r - k_r\right)r - b(\hat{\theta} - \theta)^T \Phi(x)
\end{align*}
\]  

(7.12)

Define the parameter estimation errors to be

\[
\begin{align*}
    \Delta k_x &= \hat{k}_x - k_x \\
    \Delta k_r &= \hat{k}_r - k_r \\
    \Delta \theta &= \hat{\theta} - \theta
\end{align*}
\]  

(7.13)

Then the closed-loop dynamics of the tracking error signal \( e(t) = x(t) - x_m(t) \) can be obtained by subtracting (7.6) from (7.12).

\[
\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) = a_m e + b(\Delta k_x x + \Delta k_r r - \Delta \theta^T \Phi(x))
\]  

(7.14)
Consider the Lyapunov function (candidate).

\[ V(e, \Delta k_x, \Delta k_r, \Delta \theta) = e^2 + b\left(\gamma_x^{-1} \Delta k_x^2 + \gamma_r^{-1} \Delta k_r^2 + \Delta \theta^T \Gamma_{\theta}^{-1} \Delta \theta\right) \] (7.15)

where \(\gamma_x > 0\), \(\gamma_r > 0\), and \(\Gamma_{\theta} = \Gamma_{\theta}^T > 0\) are the so-called rates of adaptation.

Taking the time derivative of \(V\) along the trajectories of (7.14), one gets

\[
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta \theta) = 2 e \dot{e} + 2 b\left(\gamma_x^{-1} \Delta k_x \dot{k}_x + \gamma_r^{-1} \Delta k_r \dot{k}_r + \Delta \theta^T \Gamma_{\theta}^{-1} \dot{\theta}\right)
\]

\[
= 2 e \left(a_m e - b\left(\Delta k_x x + \Delta k_r r - \Delta \theta^T \Phi(x)\right)\right) + 2 b\left(\gamma_x^{-1} \Delta k_x \dot{k}_x + \gamma_r^{-1} \Delta k_r \dot{k}_r + \Delta \theta^T \Gamma_{\theta}^{-1} \dot{\theta}\right)
\]

\[
= 2 a_m e^2 + 2 b\left(\Delta k_x \left(e \operatorname{sgn}(b) + \gamma_x^{-1} \dot{k}_x\right)\right) + 2 b\left|\Delta \theta^T \left(-\Phi(x) e \operatorname{sgn}(b) + \Gamma_{\theta}^{-1} \dot{\theta}\right)\right|
\] (7.16)

Using (7.16), the adaptive laws are chosen to enforce closed-loop stability.

\[
\dot{k}_x = -\gamma_x e \operatorname{sgn}(b)
\]

\[
\dot{k}_r = -\gamma_r e \operatorname{sgn}(b)
\]

\[
\dot{\theta} = \Gamma_{\theta} \Phi(x) e \operatorname{sgn}(b)
\] (7.17)

In fact, due to (7.17) the time derivative becomes negative semi-definite, that is

\[
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta \theta) = 2 a_m e(t)^2 \leq 0
\] (7.18)

which immediately implies that the signals \(e, \Delta k_x, \Delta k_r, \Delta \theta\) are uniformly bounded. The latter coupled with the fact that \(x_m(t), r(t)\) are bounded and \(\theta\) is a constant vector, implies that the system state \(x(t)\) and the estimated vector of parameters \(\hat{\theta}(t)\) are uniformly bounded. It was assumed that the vector of the components \(\varphi_i(x)\) of the regressor vector \(\Phi(x)\) were piece-wise continuous functions of \(x\). Therefore, they are uniformly bounded. Hence, the control \(u(t)\) in (7.10) is uniformly bounded. Consequently, both \(\dot{x}(t)\) and \(\dot{x}_m(t)\) are uniformly bounded.

Differentiating (7.18) yields

\[
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta \theta) = 4 a_m e(t) \dot{e}(t)
\] (7.19)
Therefore $\dot{V}$ is bounded and, consequently, $\dot{V}$ is uniformly continuous function of time. Since $V$ is lower bounded, $\dot{V}$ is negative semi-definite and uniformly continuous, then all the three conditions of the Lyapunov-like lemma (Corollary 6.2) are satisfied, and therefore

$$\lim_{t \to \infty} \dot{V}(t) = 0$$

(7.20)

Due to (7.18), we can finally conclude that the tracking error goes to zero asymptotically, as $t \to \infty$. Moreover, since the Lyapunov function is radially unbounded, the control solution is global, that is the closed-loop tracking error dynamics is globally asymptotically stable. The tracking problem is solved.

**Theorem 7.1**

For the uncertain dynamical system in (7.4) with the controller in (7.10) and the adaptive laws in (7.17), the closed-loop state $x(t)$ asymptotically tracks the state $x_m(t)$ of the reference model in (7.6), while all the signals in the closed-loop system remain bounded. Moreover, the closed-loop tracking error dynamics in (7.14) is globally asymptotically stable.

### 8. Dynamic Inversion based MRAC Design for 1st Order Systems

Using similar design approach a dynamic inversion (DI) based adaptive control laws can be derived. Consider the uncertain dynamical system

$$\dot{x} = ax + bu + f(x)$$

(8.1)

Let the constants $a$ and $b$ be unknown. Assume that $b \geq b_0 > 0$, where $b_0$ is the known lower bound of $b$. Also assume that the unknown possible nonlinear function $f(x)$ is linearly parameterized in terms of the unknown constants $\theta_i$ and known bounded basis functions $\varphi_i(x)$, that is:

$$f(x) = \sum_{i=1}^{N} \theta_i \varphi_i(x) = \theta^T \Phi(x)$$

(8.2)

Let the reference model dynamics be specified as:

$$\dot{x}_m = a_m x_m + b_m r(t)$$

$$a_m < 0, \quad |r(t)| < \infty$$

(8.3)
Rewrite the system dynamics in the form:

\[
\dot{x} = \hat{a} x + \hat{b} u + \hat{f}(x) - \left(\Delta a + \hat{a}\Delta x - b - \hat{b}\Delta u - \Delta f\right)
\]

(8.4)

where \(\hat{a}, \hat{b},\) and \(\hat{f}(x) = \theta^T \Phi(x)\) represent the on-line estimated quantities, while \(\Delta a,\) \(\Delta b,\) and \(\Delta f(x)\) are the corresponding approximation errors. Using

\[
\hat{f}(x) = \sum_{i=1}^{N} \hat{\theta}_i \phi_i(x) = \hat{\theta}^T \Phi(x)
\]

(8.5)

the function approximation error can be written as:

\[
\Delta \hat{f}(x) = \hat{f}(x) - f(x) = \sum_{i=1}^{N} \left(\hat{\theta}_i - \theta_i\right) \phi_i(x) = \Delta \theta^T \Phi(x)
\]

(8.6)

Consider the following dynamic inversion based adaptive controller

\[
u = \frac{1}{b} \left((a_m - \hat{a}) x + b_m r - \hat{\theta}^T \Phi(x)\right)
\]

(8.7)

Substituting (8.7) into the 2\(^{nd}\) term of (8.4), yields

\[
\dot{x} = a_m x + b_m r - \Delta a x - \Delta b u - \Delta \theta^T \Phi(x)
\]

(8.8)

Let \(e = x - x_m\) be the tracking error signal. Its dynamics can be obtained by subtracting (8.3) from (8.8).

\[
\dot{e} = a_m e - \Delta a x - \Delta b u - \Delta \theta^T \Phi(x)
\]

(8.9)

Consider the following Lyapunov function candidate:

\[
V(e, \Delta a, \Delta b, \Delta \theta) = e^2 + \gamma_a^{-1} \Delta a^2 + \gamma_b^{-1} \Delta b^2 + \Delta \theta^T \Gamma_0^{-1} \Delta \theta
\]

(8.10)

where \(\gamma_a > 0, \; \gamma_b > 0, \; \Gamma_0 = \Gamma_0^T > 0\) will eventually become the adaptation rates. The timed derivative of \(V\) along the trajectories of the error dynamics (8.9) can be computed:
\[ \dot{V}(e, \Delta a, \Delta b, \Delta \theta) = 2e \dot{e} + 2\left( \gamma_a^{-1} \Delta a \dot{a} + \gamma_b^{-1} \Delta b \dot{b} + \Delta \theta^T \Gamma_\theta^{-1} \dot{\theta} \right) \]

\[ = 2e\left( a_m e - \Delta a x - \Delta b u - \Delta \theta^T \Phi(x) \right) + 2\left( \gamma_a^{-1} \Delta a \dot{a} + \gamma_b^{-1} \Delta b \dot{b} + \Delta \theta^T \Gamma_\theta^{-1} \dot{\theta} \right) \]  

(8.11)

\[ = 2a_m e^2 + \Delta a\left( \gamma_a^{-1} \dot{a} - x e \right) + \Delta b\left( \gamma_b^{-1} \dot{b} - u e \right) + \Delta \theta^T \left( \Gamma_\theta^{-1} \dot{\theta} - \Phi(x) e \right) \]

Based on (8.11) and in order to make \( \dot{V} \leq 0 \), the adaptive laws are chosen as:

\[ \dot{a} = \gamma_a x e \]

\[ \dot{b} = \gamma_b u e \]

\[ \dot{\theta} = \Gamma_\theta \Phi(x) e \]  

(8.12)

In fact, this leads to

\[ \dot{V}(e, \Delta a, \Delta b, \Delta \theta) = 2a_m e^2 \leq 0 \]  

(8.13)

Therefore, the signals \( e, \Delta a, \Delta b, \Delta \theta \) are bounded. Since \( r(t) \) is bounded, then the reference model state \( x_m \) is bounded. Hence, \( x, \dot{x}, \dot{b}, \dot{\theta} \) are bounded.

Due to the division by \( \dot{b} \) in (8.7) and in order to keep the control signal \( u \) bounded, we need to modify adaptive laws (8.12). Consider the following modification of the 2nd equation in (8.12):

\[ \dot{b} = \begin{cases} 
\gamma_b u e, & \text{if } \dot{b} \geq b_0 \lor \left[ \dot{b} = b_0 \land (u e) > 0 \right] \\
0, & \text{if } \dot{b} = b_0 \land (u e) < 0 
\end{cases} \]  

(8.14)

Basically, the intent is to stop the adaptation if the \( \dot{b} \) reaches its lower limit \( b_0 \) and its time derivative is negative. One needs to verify that this modification does not adversely effects the closed-loop stability. Formally, we need to show that

\[ \Delta b \left( \gamma_b^{-1} \dot{b} - u e \right) \leq 0 \]  

(8.15)

When \( \dot{b} \geq b_0 \), the adaptive law (8.14) is the same as the corresponding law in (8.12) and, therefore \( \dot{V} \leq 2a_m e^2 \leq 0 \). Suppose that there exists \( T \) such that \( \dot{b}(T) = b_0 \). Since \( b \geq b_0 \) then \( \Delta b(T) = \dot{b}(T) - b = b_0 - b \leq 0 \). If \( u(T) e(T) \geq 0 \) then again \( \dot{V} = 2a_m e^2 \leq 0 \), while \( \dot{b}(T) = \gamma_b u e \geq 0 \) implying that \( \dot{b}(t) \) increases locally for \( t \geq T \). On other hand, if
\( u(T) e(T) < 0 \) then according to (8.14), \( \Delta b \left( y_b^{-1} \dot{b} - u e \right) = - \Delta b u e \leq 0 \). As a result, 
\( \dot{V} \leq 2 a_m e^2 \leq 0 \). Thus, modification (8.14) always contributes to making the time derivative of \( V \) to be negative-semidefinite.

The adaptive laws can now be written explicitly.

\[
\begin{align*}
\dot{\hat{a}} &= y_a x(x-x_m) \\
\dot{\hat{b}} &= y_b u(x-x_m), \quad \text{if } \hat{b} \geq b_0 \lor \left[ \hat{b} = b_0 \land (u(x-x_m)) > 0 \right] \\
\dot{\hat{\theta}} &= \Gamma_\theta \Phi(x)(x-x_m)
\end{align*}
\]

(8.16)

Next, a formal proof is given to show that the DI based adaptive control in (8.7) provides asymptotic tracking of the reference model state.

Since \( \dot{V} \leq 0 \) then \( e, \Delta a, \Delta b, \Delta \theta \) are bounded. The latter implies that \( x, \dot{x}, \hat{b}, \hat{\theta} \) are bounded. Due to modification (8.14), \( \hat{b} \geq b_0 \) and consequently \( u \) is bounded. This in turn implies that \( \dot{x} \) is bounded. Moreover, since \( r \) is bounded, then \( \dot{x}_m \) is bounded and therefore \( \dot{e} \) is bounded. Because of (8.14)

\[
\dot{V}(e, \Delta a, \Delta b, \Delta \theta) \leq -2 |a_m| e^2 \leq 0
\]

(8.17)

for all \( t \geq 0 \). Since \( V \) is bounded from below by zero and its derivative is semi-negative, \( V \) converges to a limit, as a function of time. Integrating both sides of (8.17) yields:

\[
V(t) - V(0) \leq -2 |a_m| \int_0^t e^2(\tau) d\tau \leq 0
\]

(8.18)

or, equivalently:

\[
\int_0^t e^2(\tau) d\tau \leq \frac{1}{2 |a_m|} (V(0) - V(t)) < \infty
\]

(8.19)

Let \( W(t) = \int_0^t e^2(\tau) d\tau \). From (8.19) it follows that \( W(t) \) tends to a finite limit, as \( t \to \infty \).

At the same time its time derivative is \( \dot{W}(t) = e^2(t) \). Since \( \dot{W}(t) = 2 e \dot{e}(t) < \infty \) then
$\dot{W}(t)$ is uniformly continuous. Using Barbalat’s Lemma, implies that $\lim_{t \to \infty} \dot{W}(t) = 0$.
Thus, $\lim_{t \to \infty} e^2(t) = 0$ and the tracking problem is solved.

**Remark 8.1**
Modification (8.14) is a special case of the well-known Projection Operator. Since the right hand side of (8.14) is not Lipschitz the closed-loop system does not satisfy the sufficient conditions to have a unique trajectory, given an initial state. Corresponding solutions can be defined similar to the case of variable structure systems such as systems with sliding modes. Nevertheless, a continuous version of the Projection Operator exists and will be covered later in the course.

## 9. MRAC Design for Affine-in-Control MIMO Systems

**Reading material:**
[1]: Chapter 8, Section 8.3
[1]: Chapter 8, Section 8.5

In this section, we consider MRAC design for a class of multi-input-multi-output (MIMO) nonlinear systems whose plant dynamics is linearly parameterized, the uncertainties satisfy the so-called matching conditions, and if the full state is measurable, (i.e., available on-line as the system output). More specifically, consider the $n^{th}$ order MIMO system in the form:

$$
\dot{x} = Ax + B\Lambda(u + f(x))
$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $B \in \mathbb{R}^{m \times m}$ is known matrix, $A \in \mathbb{R}^{n \times n}$ and $\Lambda \in \mathbb{R}^{m \times m}$ are unknown matrices. In addition, it is assumed that $\Lambda$ is diagonal, its elements $\lambda_i$ are non-negative, and the pair $(A, B\Lambda)$ is controllable. The uncertainty in $\Lambda$ is introduced to model a control failure phenomenon.

Moreover, the unknown possibly nonlinear function $f(x): \mathbb{R}^n \to \mathbb{R}^m$ represents the so-called *system matched uncertainty*. It is assumed that the function can be written as a linear combination of $N$ known *bounded* basis functions with unknown constant coefficients.

$$
f(x) = \Theta^T \Phi(x)
$$

In (9.2), $\Theta \in \mathbb{R}^{N \times m}$ is the unknown constant matrix, while $\Phi(x) \in \mathbb{R}^N$ represents the known regressor vector.
The **control objective** of the **MIMO tracking problem** is to choose the input vector $u$ such that all signals in the closed-loop system are bounded and the state $x$ follows the state $x_{ref} \in \mathbb{R}^n$ of a reference model specified by the LTI system

$$
\dot{x}_{ref} = A_{ref} x_{ref} + B_{ref} r(t)
$$

(9.3)

where $A_{ref} \in \mathbb{R}^{n \times n}$ is Hurwitz, $B_{ref} \in \mathbb{R}^{n \times m}$, and $r(t) \in \mathbb{R}^m$ is a bounded reference input vector. Note that the reference model and its external input $r(t)$ must be chosen so that $x_{ref}(t)$ represents a desired trajectory that $x(t)$ has to follow. In other words, the control input $u$ needs to be chosen such that the tracking error vector asymptotically tends to zero.

$$
\lim_{t \to \infty} \left\| x(t) - x_{ref}(t) \right\| = 0
$$

(9.4)

If the matrices $A$ and $\Lambda$ were known, one could apply the control law

$$
u = K_x^T x + K_r^T r - \Theta^T \Phi(x)
$$

(9.5)

and obtain the closed-loop system

$$
\dot{x} = \left( A + B \Lambda K_x^T \right) x + B \Lambda K_r^T r
$$

(9.6)

Comparing (9.6) with the desired dynamics in (9.3), it follows that the ideal (unknown) matrix gains must be chosen to satisfy the so-called **matching conditions**:

$$
\begin{bmatrix}
A + B \Lambda K_x^T = A_{ref} \\
B \Lambda K_r^T = B_{ref}
\end{bmatrix}
$$

(9.7)

Assuming that the matching conditions take place, it is easy to see that the closed-loop system is the same as that of the reference model, and consequently, asymptotic (exponential) tracking is achieved for any bounded reference input signal $r(t)$.

**Remark 9.1**

Given the matrices $A$, $B$, $\Lambda$, $A_{ref}$, $B_{ref}$, no $K_x$, $K_r$ may exist to satisfy the matching conditions (9.7) indicating that the control law (9.5) may not have enough structural flexibility to meet the control objective. Often in practice, the structure of $A$ is known, and the reference model matrices $A_{ref}$, $B_{ref}$ are chosen so that (9.7) has a solution for $K_x$, $K_r$. 