Lecture 3

Let $\Omega \subset R^n$ denote a bounded domain and suppose that the system dynamics are:

$$\dot{x} = A\,x + B\frac{\varepsilon(t,x)}{2} \tag{5.5}$$

where A is Hurwitz. Also suppose that $\varepsilon(t,x) \in R^m$ is bounded on Ω. Let $Q = Q^T > 0$ and consider

$$V(x) = x^T\,P\,x \tag{5.6}$$

where $P = P^T > 0$ is the unique positive definite symmetric solution of the algebraic Lyapunov equation.

$$P\,A + A^T\,P = -Q \tag{5.7}$$

Then the time derivative of V evaluated along the system **(5.1)** trajectories satisfies the following relation:

$$\dot{V}(x) = -x^T\,Q\,x + x^T\,P\,B\,\varepsilon(t,x), \quad \forall\, x \in \Omega, \quad \forall\, t \geq t_0 \tag{5.8}$$

Suppose that $R > 0$ is chosen such that the sphere S_R is inside the domain Ω, that is:

$$S_R \triangleq \{\|x\| \leq R\} \subset \Omega \tag{5.9}$$

Also suppose that $|\varepsilon(t,x)| \leq \varepsilon_{\max} < \infty$ for all $x \in S_R$, uniformly in t. Then one can formally prove that all the solutions $x(t)$ of (5.1) that start in a subset of S_R are UUB.

We start our formal arguments with the well-known double-inequality, which is valid for any positive definite matrix P and for all vectors x.

$$\lambda_{\min}(P)\|x\|^2 \leq x^T\,P\,x \leq \lambda_{\max}(P)\|x\|^2 \tag{5.10}$$

where $\lambda_{\min}(P), \lambda_{\max}(P)$ denote the smallest and the largest eigenvalues of P, respectively.

An upper bound for $\dot{V}$ in (5.8) can be calculated, for $x \in \Omega$.

$$\dot{V}(x) \le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|P\,B\|\varepsilon_{\max} = -\|x\|\left(\lambda_{\min}(Q)\|x\| - \|P\,B\|\varepsilon_{\max}\right) \tag{5.11}$$

Define sphere $S_r \triangleq \left\{\|x\| \le r \triangleq \dfrac{\|P\,B\|\varepsilon_{\max}}{\lambda_{\min}(Q)}\right\}$. Then it follows from (5.11) that

$$\dot{V}(x) < 0, \quad \forall\, x \in \Lambda = \left\{r \le \|x\| \le R\right\} = S_R - S_r \tag{5.12}$$

Let $b_r = \lambda_{\max}(P)r^2$ and let $\Omega_{b_r} \triangleq \left\{V(x) \le b_r\right\}$. Then $S_r \subset \Omega_{b_r}$. In fact, if $x \in S_r$ then using the right hand side of (5.10) yields:

$$x^T\,P\,x \le \lambda_{\max}(P)\|x\|^2 \le \lambda_{\max}(P)r^2 = b_r \tag{5.13}$$

Hence, $x \in \Omega_{b_r}$ and the inclusion $S_r \subset \Omega_{b_r}$ is proven.

Let $b_R = \lambda_{\min}(P)R^2$ and define $\Omega_{b_R} \triangleq \left\{V(x) \le b_R\right\}$. Then $\Omega_{b_R} \subset S_R$. In fact, if $x \in \Omega_{b_R}$ then using the left hand side of (5.10), yields:

$$\lambda_{\min}(P)\|x\|^2 \le x^T\,P\,x = V(x) \le b_R = \lambda_{\min}(P)R^2 \tag{5.14}$$

Hence, $\|x\| \le R$, that is $x \in S_R$, and the inclusion $\Omega_{b_R} \subset S_R$ is proven.

Next, we need to ensure that $b_r < b_R$. The latter implies

$$b_r = \lambda_{\max}(P)r^2 < \lambda_{\min}(P)R^2 = B_R \tag{5.15}$$

or, equivalently:

$$\frac{r}{R} < \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \tag{5.16}$$

The above inequality can be viewed as a restriction on the eigenvalues of P and the constants r and R. This relation ensures that

$$S_r \subset \Omega_{b_r} \subset \Omega_{b_R} \subset S_R \tag{5.17}$$

Graphical representation of the four sets is given in Figure 5.2.

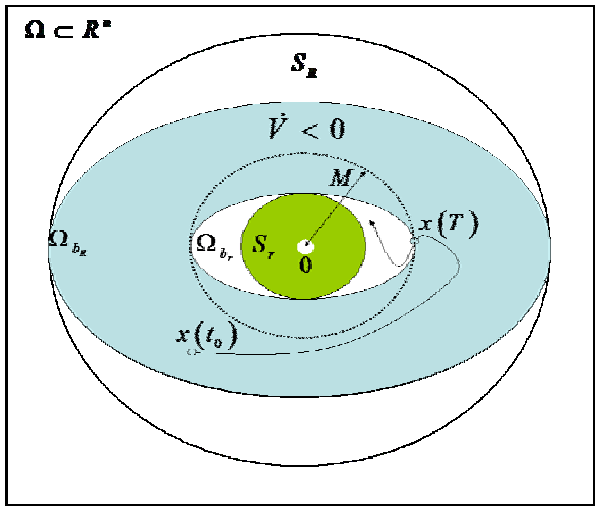

Figure 5.2: Representation of the sets $S_r \subset \Omega_{b_r} \subset \Omega_{b_R} \subset S_R$ and the Ultimate Bound M

Next, we show that all solutions starting in Ω_{b_R} will enter Ω_{b_r} and remain there afterwards.

If $x(t_0) \in \Omega_{b_r}$ then since $\dot{V} < 0$ in $\Lambda = \Omega_{b_R} - \Omega_{b_r}$, $V(x(t))$ is a decreasing function of time outside of Ω_{b_r}. Therefore, solutions that start in Ω_{b_r} will remain there.

Suppose that $x(t_0) \in \Lambda$. Inequality (5.11) implies that

$$\begin{aligned}
\dot{V}(x) &\le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|PB\|\varepsilon_{\max} \\
&= -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\underbrace{\lambda_{\max}(P)\|x\|^2}_{\ge V(x)} + \frac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}\underbrace{\sqrt{\lambda_{\min}(P)}\|x\|}_{\le\sqrt{V(x)}} \\
&\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(x) + \frac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}\sqrt{V(x)}
\end{aligned} \tag{5.18}$$

Thus, $V(x(t)) \ge 0$ satisfies the following differential inequality, (as a function of time):

$$\dot{V}(x) \le -aV(x) + g\sqrt{V(x)} \tag{5.19}$$

where $a = \dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and $g = \dfrac{\|PB\|\varepsilon_{\max}}{\sqrt{\lambda_{\min}(P)}}$ are positive constants. Let $W(x) = \sqrt{V(x)}$. Then relation (5.19) is equivalent to

$$\dot{W}(x) \le -\frac{a}{2} W(x) + \frac{g}{2} \tag{5.20}$$

Define

$$Z(x) \triangleq \dot{W}(x) + \frac{a}{2} W(x) \tag{5.21}$$

Inequality (5.20) implies that

$$Z(x(t)) \le \frac{g}{2}, \quad \forall t \ge t_0 \tag{5.22}$$

Solving (5.21) for W yields:

$$W(x(t)) = W(x(t_0)) e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} Z(x(\tau)) d\tau \tag{5.23}$$

Therefore

$$\begin{aligned} W(x(t)) &\le W(x(t_0)) e^{-\frac{a}{2}(t-t_0)} + \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} \left| Z(x(\tau)) \right| d\tau \\ &\le W(x(t_0)) e^{-\frac{a}{2}(t-t_0)} + \frac{g}{2} \int_{t_0}^{t} e^{-\frac{a}{2}(t-\tau)} d\tau = W(x(t_0)) e^{-\frac{a}{2}(t-t_0)} + \frac{g}{a}\left[1 - e^{-\frac{a}{2}(t-t_0)}\right] \\ &= \frac{g}{a} + \underbrace{e^{-\frac{a}{2}(t-t_0)}\left[W(x(t_0)) - \frac{g}{a}\right]}_{\mathrm{o}(1)} = \frac{g}{a} + \mathrm{o}(1) \end{aligned} \tag{5.24}$$

As $t \to \infty$ and in terms of the original variables, one gets

$$\sqrt{x^T(t) P x(t)} \le \frac{g}{a} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \underbrace{\left(\frac{\|PB\| \varepsilon_{\max}}{\lambda_{\min}(Q)} \right)}_{r} + \mathrm{o}(1) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} r + \mathrm{o}(1) \tag{5.25}$$

Choose $\delta > 0$. Then it is easy to see that there exists T independent of t_0 such that $\mathrm{o}(1) \le \delta$ and , consequently

$$\sqrt{x^T(t) P x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} r + \delta, \quad \forall t \ge T + t_0, \quad \forall x \in \Omega_{b_R} \tag{5.26}$$

Since the above relation is valid for all solutions that start in Λ, it is also valid for the solution which starts in Λ and maximizes the left hand side of the inequality. In other words

$$\max_{x\in\Omega_B}\sqrt{x^T(t)\,P\,x(t)} \le \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\,r+\delta \tag{5.27}$$

On the other hand

$$\max_{x\in\Omega_B}\sqrt{x^T(t)\,P\,x(t)} = \sqrt{\lambda_{\max}(P)}\max_{x\in\Omega_B}\|x(t)\| \tag{5.28}$$

Hence

$$\|x(t)\| \le M \triangleq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,r+\frac{\delta}{\sqrt{\lambda_{\max}(P)}},\quad \forall\, t\ge T+t_0,\quad \forall\, x\in\Omega_{b_R} \tag{5.29}$$

Consequently, the solutions of **(5.1)** are UUB with the ultimate bound M. The bound is shown in Figure 5.2. Summarizing the results, we state the following theorem.

Theorem 5.1
Let $P=P^T>0$ and $Q=Q^T>0$ be symmetric positive definite matrices that satisfy the Lyapunov algebraic equation (5.7), and let $V(x)=x^T\,P\,x$. Consider the dynamics in (5.5) and suppose that $|\varepsilon(t,x)|\le\varepsilon_{\max}<\infty$, uniformly in t, and for all $x\in S_R=\{\|x\|\le R\}$, where $R>0$ is chosen such that $S_R\triangleq\{\|x\|\le R\}\subset\Omega$. Let $S_r\triangleq\left\{\|x\|\le r\triangleq\frac{\|P\,B\|\,\varepsilon_{\max}}{\lambda_{\min}(Q)}\right\}$ and suppose that $\frac{r}{R}<\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$. Then those solutions of (5.5) that start in the bounded set $\Omega_{b_R}\triangleq\{V(x)\le b_r=\lambda_{\max}(P)\,r^2\}$ are UUB, with the ultimate bound M, as it is defined in (5.29).

6. Invariance – like Theorems

Reading material:
[1]: Section 4.5
[2]: Section 8.3

For autonomous systems, LaSalle's invariance set theorems allow asymptotic stability conclusions to be drawn even when $\dot V$ is only negative semi-definite in a domain Ω. In

that case, the system trajectory approaches the largest invariant set E, which is a subset of all points $x \in \Omega$ where $\dot{V}(x) = 0$. However the invariant set theorems are not applicable to nonautonomous systems. In the case of the latter, it may not even be clear how to define a set E, since V may explicitly depend on both t and x. Even when $V = V(x)$ does not explicitly depend on t the nonautonomous nature of the system dynamics precludes the use of the LaSalle's invariant set theorems.

Example 6.1
The closed-loop error dynamics of an adaptive control system for 1st order plant with one unknown parameter is

$$\dot{e} = -e + \theta\, w(t)$$
$$\dot{\theta} = -e\, w(t)$$

where e represents the tracking error and $w(t)$ is a bounded function of time t. Due to the presence of $w(t)$, the system dynamics is nonautonomous. Consider the Lyapunov function candidate

$$V(e,\theta) = e^2 + \theta^2$$

Its time derivative along the system trajectories is

$$\dot{V}(e,\theta) = 2e\dot{e} + 2\theta\dot{\theta} = 2e\big(-e + \theta\, w(t)\big) + 2\theta\big(-e\, w(t)\big) = -2e^2 \le 0$$

This implies that V is a decreasing function of time, and therefore, both $e(t)$ and $\theta(t)$ are bounded signals of time. But due to the nonautonomous nature of the system dynamics, the LaSalle's <u>*invariance set theorems cannot be used*</u> to conclude the convergence of $e(t)$ to the origin.

In general, if $\dot{V}(t,x) \le -W(x) \le 0$ then we may expect that the trajectory of the system approaches the set $\{W(x) = 0\}$, as $t \to \infty$. Before we formulate main results, we state a lemma that is interesting in its own sake. The lemma is an important result about asymptotic properties of functions and their derivatives and it is known as the Barbalat's lemma. We begin with the definition of a uniform continuity.

Definition 6.1 (uniform continuity)
A function $f(t): R \to R$ is said to be <u>*uniformly continuous*</u> if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \quad \forall |t_2 - t_1| \le \delta \Rightarrow \big| f(t_2) - f(t_1) \big| \le \varepsilon$$

Note that t_1 and t_2 play a symmetric role in the definition of the uniform continuity.

Lemma 6.1 (Barbalat)
Let $f(t): R \to R$ be differentiable and has a finite limit as $t \to \infty$. If $\dot{f}(t)$ is uniformly continuous then $\dot{f}(t) \to 0$ as $t \to \infty$, (see proof in [2], p. 323).

Lemma 6.2
If $\dot{f}(t)$ is bounded then $f(t)$ is uniformly continuous.

An immediate and practical corollary of Barbalat's lemma can now be stated.

Corollary 6.1
If $f(t): R \to R$ is twice differentiable, has a finite limit, and its 2nd derivative is bounded then $\dot{f}(t) \to 0$ as $t \to \infty$.

In general, the fact that derivative tends to zero does not imply that the function has a limit. Also, the converse is not true. In other words:

$$f(t) \to C \not\Rightarrow \dot{f}(t) \to 0$$

Example 6.2

- As $t \to \infty$, $f(t) = \sin(\ln t)$ does not have a limit, while $\dot{f}(t) = \dfrac{\cos(\ln t)}{t} \to 0$.
- As $t \to \infty$, $f(t) = e^{-t}\sin\left(e^{2t}\right) \to 0$, while $\dot{f}(t) = -e^{-t}\sin\left(e^{2t}\right) + e^{t}\sin\left(e^{2t}\right) \to \infty$.

Example 6.3
Consider an LTI system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A and a uniformly bounded in time input $u(t)$. These two facts imply that the state $x(t)$ is bounded. Thus, the state derivative $\dot{x}(t)$ is bounded. Let $y = Cx$ represent the system output. Then $\dot{y} = C\dot{x}$ is bounded and, consequently the system output $y(t)$ is a <u>*uniformly continuous*</u> function of time. Moreover, if the input $u(t) = u_0$ is constant then the output $y(t)$ tends to a limit, as $t \to \infty$. The latter combined with the fact that y is uniformly continuous implies, (by Barbalat's Lemma), that the output time derivative $\dot{y}$ asymptotically approaches zero.

To apply Barbalat's lemma to the analysis of nonautonomous dynamic systems we state the following immediate corollary.

Corollary 6.2 (Lyapunov-like Lemma)
If a scalar function $V = V(t,x)$ is such that

- $V(t,x)$ is lower bounded
- $\dot{V}(t,x)$ is negative semi-definite along the trajectories of $\dot{x} = f(t,x)$
- $\dot{V}(t,x)$ is uniformly continuous in time

then $\dot{V}(t,x) \to 0$, as $t \to \infty$.

Notice that the first two assumptions imply that $V(t,x)$ tends to a limit. The latter coupled with the 3rd assumption proves (using Barbalat's lemma) the corollary.

Example 6.4
Consider again the closed-loop error dynamics of an adaptive control system from Example 6.1. Choosing $V(e,\theta) = e^2 + \theta^2$, it was shown that along the system trajectories: $\dot{V}(e,\theta) = -2e^2 \le 0$. The 2nd time derivative of V is

$$\ddot{V}(e,\theta) = -4e\dot{e} = -4e\left(-e + \theta w(t)\right)$$

Since $w(t)$ is bounded by hypothesis, and $e(t)$ and $\theta(t)$ were shown to be bounded, it is clear that $\ddot{V}$ is bounded.. Hence, $\dot{V}$ is uniformly continuous and by the Barbalat's lemma (or the Lyapunov-like lemma), $\dot{V} \to 0$ which in turn indicates that the tracking error $e(t)$ tends to zero, as $t \to \infty$.

7. Basic Concepts and Introduction to Adaptive Control

Reading material:
[1]: Chapter 8, Section 8.1
[2]: Section 1.2.6
[2]: Section 4.2, Example 4.10
[2]: Section 12.1

Introduction

Since the 1950's adaptive control has firmly remained in the mainstream of controls and dynamics research, and it has grown to become a well-formed scientific discipline. One of the reasons for the continuing popularity and rapid growth of adaptative control is its clearly defined goal – to control dynamical systems with unknown parameters.

Research in adaptive control started in connection with the design of autopilots for high-performance aircraft. But interest in the subject has soon diminished due to the lack of insights and the crash of a test flight, (NASA X-15 program). The last decade has witnessed the development of a coherent theory for adaptive control, which has lead to many practical applications in the areas such as aerospace, robotics, chemical process control, ship steering, bioengineering, and many others.

The basic idea in adaptive control is to estimate the uncertain plant and / or controller parameters on-line based on the measured system signals and use the estimated parameters in control input computation. An adaptive controller can thus be regarded as an inherently nonlinear dynamic system with on-line parameter estimation.

Generally speaking, *the basic objective* of adaptive control is to maintain consistent performance of a system in the presence of uncertainty or unknown variation in plant parameters.

There are two main approaches for constructing adaptive controllers:

- Model reference adaptive control (MRAC) method
- Self-tuning control (STC) method

Schematic representation of an MRAC system is given in Figure 7.1.

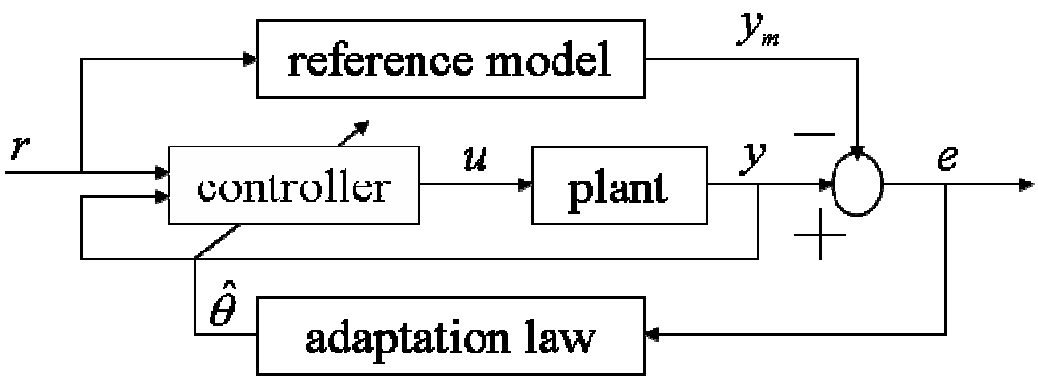

Figure 7.1: Model Reference Adaptive Control System

The MRAC system is composed of *four parts*:

- Plant of a known structure but with unknown parameters
- Reference model for specification of the desired system output
- Feedback / feedforward control law with adjustable gains, (controller)
- Parameter / gain adaptation law

Schematic representation of an STC system is given in Figure 7.2.

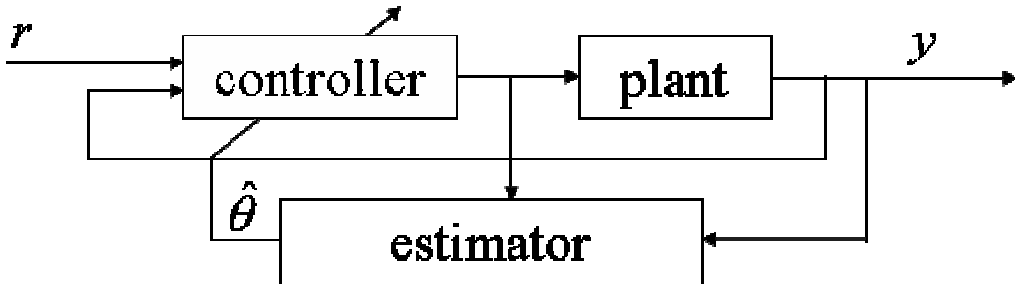

Figure 7.2: Self-Tuning Control System

The STC system combines a controller with an on-line (recursive) plant parameter estimator. A reference model can be added to the architecture. Basically, STC system performs simultaneous parameter identification and control. The controller parameters are computed from the estimates of the unknown plant parameters, as they were the true ones. This idea is often referred to as the *Certainty Equivalence Principle*. By coupling different control and estimation schemes, one can obtain a variety of self-tuning regulators.

When the true plane parameters are unknown, the controller parameters are either estimated directly (*direct* schemes) or computed by solving the same design equations using plant parameter estimates (*indirect* schemes). MRAC and STC systems can be designed using both direct and indirect approaches.

Our focus will be on the design, analysis and evaluation of the direct MRAC systems for continuous plants with uncertain dynamics.

Tracking Control Problem

In particular, we consider tracking problems for continuous plants operating in the presence of modeling uncertainties, environmental disturbances, and control failures. State feedback / feedforward tracking control will be designed for uncertain dynamical systems in the form

$$\begin{aligned} \dot{x} &= f\left(t, x, u, \Theta\right) \\ y &= h\left(x, \Theta\right) \end{aligned} \tag{7.1}$$

where x is the state, u is the control, Θ is a vector of unknown constant parameters, y is the controlled output. It is assumed that the system state vector x is available (measured on-line).

The tracking problem is to design the control input u so that the controlled output $y(t)$ tracks a given reference signal $r(t)$ in the presence of the system uncertainties, that is the *output tracking error*

$$e_y(t) = y(t) - r(t) \tag{7.2}$$

becomes sufficiently small, as $t \to \infty$. Moreover, it is required that during tracking, all the signals in the corresponding closed-loop system remain bounded.

If $e_y(t) \to 0$ then we say that an <u>*asymptotic output tracking*</u> is achieved. In general, it might not be feasible to achieve asymptotic tracking. In that case, the goal will be to achieve ultimate boundedness of the tracking error within a prescribed tolerance, that is

$$\left\| e_y(t) \right\| \le \varepsilon, \quad \forall t \ge T \tag{7.3}$$

where ε is the prescribed small positive number.

References

1. J.J. Slotine, W. Li, Applied Nonlinear Control, Prentice Hall, 1995.
2. H.K. Khalil, Nonlinear Systems, 3rd Edition, Prentice Hall, New Jersey, 2000.