Normal Forms Theory

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1 Normal Form Theory

Introduction. To find a coordinate system where the dynamical system take the "simplest" form.

- The method is local in the sense that the coordinate transforms are generated near a know solution, such as a fixed point.
- The coordinate transformation will be nonlinear, but these transformation are found by solving a sequence of linear problem.
- The structure of the normal form is determined entirely by the nature of the linear part of the problem.

Preliminary Preparation. Consider

$$\dot{w} = G(w)$$

where $w \in R^n$, G is C^r, and the system has a fixed point at $w = w_0$. Then it can be written as (*)

$$\dot{x} = Jx + F(x) = Jx + F_2(x) + F_3(x) + \cdots + F_{r-1}(x) + O(|x|^r)$$

where $F_i(x)$ represent the order i terms in the Taylor expansion of $F(x)$.

Simplification of the Second Order Terms. Introduce the coordinate transformation

$$x = y + h_2(y)$$

where $h_2(y)$ is second order in y. If $h_2(y)$ can be found to satisfy the following homological equation

$$\frac{\partial h_2}{\partial y} Jy - Jh_2(y) = F_2(y)$$

then the $F_2(y)$ can be eliminated.

Simplification of the Third Order Terms. Similar computation can be done to simplify the third order terms.

Theorem 13.1 (Poincaré) If the eigenvalues of the matrix J are non-resonant, then the equation

$$\dot{x} = Jx + F(x) + \dots$$

can be transformed into the linear equation

$$\dot{y} = Jy$$

by the formal transformation $x = y + h_2(y) + \dots$

Definition: The eigenvalues $\lambda_1, \dots, \lambda_n$ of A are *resonant* if one has

$$\lambda_i = \sum_{j=1}^{n} m_j \lambda_j$$

with $m_j \in \{0\} \cup \mathbb{N}$ and $m = m_1 + m_2 + \dots + m_n \geq 2$.

Finding Solution of Homological Equation for h_2. Notice that

$$H_2 = L_J^{(2)}(H_2) \oplus G_2$$

where G_2 represent a space complementary to $L_J^{(2)}(H_2)$. We can choose $h_2(y)$ such that only second order terms that are in G_2 remain. We denote these terms by

$$F_2^r(y) \in G_2.$$

Normal Form Theorem: By a sequence of analytic coordinate changes equation (*) can be transformed into equation (**)

$$\dot{y} = Jx + F_2^r(x) + F_3^r(x) + \cdots + F_{r-1}^r(x) + O(|x|^r)$$

where $F_k^r(y) \in G_k, 2 \leq k \leq (r-1)$, and G_k is a space complementary to $L_J^{(k)}(H_k)$. Equation (**) is said to be in normal form through order $(r-1)$.

Remarks:

1. The terms $F_k^r(y), 2 \leq k \leq (r-1)$ are referred to as resonance terms.
2. The structure of the nonlinear terms in (**) is determined entirely by the linear part (i.e., J).
3. Notice that in simplifying the terms of order k, any lower order terms do not get modified. However, terms of order higher than k are modified.

Example 13.4: Find the normal form of

$$\begin{aligned}\dot{x}_1 &= 2x_1 + a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + \ldots \\ \dot{x}_2 &= x_2 + b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + \ldots\end{aligned}$$

Example 13.5: Find the normal form of a perturbed harmonic oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 + \ldots \\ \dot{x}_2 &= -x_1 + \ldots\end{aligned}$$

Remarks: (1) If J has multiple eigenvalues, the treatment given here carries through. But multiple eigenvalues do produce more resonances. (2) If one of the eigenvalues is zero, we have resonance.

Example 2.1.2. Taken-Bogdanov Normal Form.

Theorem 13.2: Consider equation $\dot{x} = Ax + f(x)$. If all the eigenvalues of A are lying either to the right or to the left of the imaginary axis in $\mathbb{C}$, the equation can be reduced to a polynomial normal form by a formal transformation.

Remark: The conclusion of Theorem 13.2 will still hold if zero is not contained in the convex hull of the collection of eigenvalues.

Normal Form for Hopf Bifurcation.