Theory of Stability for Equilbrium Solutions

CDS140A Lecturer: W.S. Koon

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1 Stability of Equilibrium Solutions

Definition (Lyapunov Stability) An equilibrium solution $\bar{x}$ is *stable* if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that for any other solution $x(t)$ with $||x(t_0) - \bar{x}|| < \delta$, we have $||x(t) - \bar{x}|| \leq \epsilon$ for $t \geq t_0$. If an equilibrium solution is not stable, it is *unstable*.

Definition (Asymptotic Stability) An equilibrium solution $\bar{x}$ is *asymptotically stable* if

- it is stable and
- for any other solution $x(t)$, there exists a $\delta > 0$ such that, if $||x(t_0) - \bar{x}|| \leq \delta$, then $\lim_{t\to\infty} ||x(t) - \bar{x}|| = 0$.

Remarks:

- These definitions require the infinite time existence of solutions.
- There exists equilibrium solutions that can attract a neighborhood, but are not Lyapunov stable.

Example: Consider

$$\begin{aligned} \dot{x} &= x - rx - ry + xy \\ \dot{y} &= y - ry + rx - x^2 \end{aligned}$$

which can be rewritten in polar coordinates as

$$\begin{aligned} \dot{r} &= r(1 - r) \\ \dot{\theta} &= r(1 - \cos\theta). \end{aligned}$$

2 Linearization

Theorem 2.1 *Suppose all of the eigenvalues of $Df(\bar{x})$ have negative real parts. Then the equilibrium solution $x = \bar{x}$ of the nonlinear equation is asymptotically stable.*

Theorem 2.2 *If any of the eigenvalues of $Df(\bar{x})$ has a positive real part, then the equilibrium solution $x = \bar{x}$ of the nonlinear equation is unstable.*

Remark: If the eigenvalues of the linearized equation have nonzero real parts, then the orbit structure near an equilibrium solution of the nonlinear equation is essentially the same as that of the linearized system.

Unforced Duffing Oscillator Consider

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 - \delta y, \qquad \delta \geq 0. \end{aligned}$$

Routh-Hurwitz Test All of the roots of the polynomial have real parts strictly less than zero iff all $(n+1)$ elements on the first column of the Routh table are nonzero and have the same sign.

$$(r_{i,1} \quad r_{i,2} \quad \cdots) = (r_{i-2,2} \quad r_{i-2,3} \quad \cdots) - \frac{r_{i-2,1}}{r_{i-1,1}}(r_{i-1,2} \quad r_{i-1,3} \quad \cdots), \qquad i > 2.$$

Example: Consider

$$x^3 + 6x^2 + 11x + 6 = 0.$$

Definitions

- A hyperbolic fixed point $\bar{x}$ is called a *saddle* if some, but not all, of the eigenvalues of the linearized system have real parts greater than zero and the rest have negative real part.
- If all of the eigenvalues have negative real part, then it is called *stable node* or *sink*.
- If all of the eignevalues have positive real part, then it is called *unstable node* or *source*.

3 Lyapunov Functons

Roughly speaking, $\bar{x}$ is stable if one can find a neighborhood U of $\bar{x}$ for which orbits starting in U remain in U for all positive times.

- The coniditon would be satisfied if the vector field is either tangent to the boundary of U or pointing inward toward $\bar{x}$
- This situation should remain true even as one shrinks U down to $\bar{x}$.

Theorem 3.1 *Let E be an open set of $\mathbb{R}^n$ containing an equilibrium point $\bar{x}$. Suppose there exists a function $V \in C^1(E)$ with $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$. Then*

- *if $L_t V \leq 0$ for all $x \in E$, $\bar{x}$ is stable;*
- *if $L_t V < 0$ for all $x \in E \backslash \{\bar{x}\}$, $\bar{x}$ is aymptotically stable;*
- *if $L_t V > 0$ for all $x \in E \backslash \{\bar{x}\}$, $\bar{x}$ is unstable.*

Example (Aymptotically Stable but Not Sink): Consider

$$\begin{aligned}
\dot{x} &= -2y + yz - x^3 \\
\dot{y} &= x - xz - y^3 \\
\dot{z} &= xy - z^3
\end{aligned}$$

Theorem 3.2 *Consider Hamiltonian equations with $(q,p) = (0,0)$ as an equilibrium solution. If $H(q,p) - H(0,0)$ is sign definite in a neighborhood of $(q,p) = (0,0)$, the equilibrium solution is stable.*

Example: Consider

$$\ddot{x} + f(x) = 0$$

where $xf(x) > 0$ for $x \neq 0$.