Stable and Unstable Invariant Manifolds

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1 Introduction

Definition: A *manifold* is a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph.

Definition: A set S is *invariant* w.r.t. a nonlinear system if for any $x_0 \in S$ we have $x(t; 0, x_0) \in S$ for all t.

Example: Consider

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y + x^2 \\ \dot{z} &= z + x^2 \end{aligned}$$

2 Local Stable, Unstable, and Center Manifolds of Fixed Points

Let $\bar{x}$ be an equilibrium (or fixed) point,

- Transform the fixed point to the origin via $y = x - \bar{x}$ and Taylor expand, we obtain

$$\dot{y} = Df(\bar{x})y + R(y).$$

- Block diagonalize $A = Df(\bar{x})$ via $T^{-1}y = (u, v, w)$, we obtain

$$\begin{aligned} \dot{u} &= A_s u + R_s(u, v, w) \\ \dot{v} &= A_u v + R_u(u, v, w) \\ \dot{w} &= A_c w + R_c(u, v, w) \end{aligned}$$

Theorem 2.1 *Suppose the system is $C^r, r \geq 2$.*

- *Then the fixed point $(u, v, w) = 0$ possess a C^r s-dimensional local, invariant stable manifold, $W^s_{loc}(0)$, a C^r u-dimensional local, invariant stable manifold, $W^u_{loc}(0)$. and a C^r c-dimensional local, invariant stable manifold, $W^c_{loc}(0)$, all intersecting at $(u, v, w) = 0$.*

- *These manifolds are all tangent to the respective invariant subspace of the associated linearized system at the origin, and, hence, are locally represented as graphs.*

- *In particular, we have*

$$\begin{aligned} W^s_{loc}(0) &= \{(u, v, w) | v = h^s_v(u), w = h^s_w(u); Dh^s_v(0) = 0 = Dh^s_w(0)\} \\ W^u_{loc}(0) &= \{(u, v, w) | u = h^u_u(v), w = h^u_w(v); Dh^u_u(0) = 0 = Dh^u_w(0)\} \\ W^c_{loc}(0) &= \{(u, v, w) | u = h^c_u(w), v = h^c_v(w); Dh^c_u(0) = 0 = Dh^c_v(0)\} \end{aligned}$$

  where the h's are C^r functions.

- *Moreover, trajectories in $W^s_{loc}(0)$ and $W^u_{loc}(0)$ have the same asymptotic properties as trajectories in E^s and E^u, respectively. Namely, trajectories with initial conditions in $W^s_{loc}(0)$ (resp., $W^u_{loc}(0)$) approach the origin at an exponential rate asymptotically as $t \to +\infty$ (resp., $t \to -\infty$).*

Remarks:

- If the fixed point is hyperbolic, i.e., $E^c = \emptyset$, trajectories of the nonlinear system locally behave the same as trajectories of the associated linear system.

- In general, the behavior of trajectories in $W^c_{loc}(0)$ cannot be inferred from the behavior of trajectories in E^c. While stable and unstable manifolds are unique, the center manifold is a bit more delicate.

- In numerical calculations of stable and unstable manifolds, it is convenient to start in a neighborhood of the saddle in points of E_s and E_u, which have been obtained from the linear analysis.

3 Computing Invariant Manifolds Using Taylor Expansion

Given the vector field

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

and a surface in the phase space represented by the graph of a function

$$y = h(x).$$

The surface is invariant if the vector field is tangent to the surface, i.e.,

$$g(x, h(x)) = Dh(x) f(x, h(x)).$$

Example: Consider

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y + x^2. \end{aligned}$$

4 Evolution of a Volume Element and Liouville's Theorem

Lemma: Consider $\dot{x} = f(x), x \in \mathbb{R}^n$, and suppose that it generates a flow $\phi_t(\cdot)$. Let D_0 denote a domain in $\mathbb{R}^n$ and let $D_t = \phi_t(D_0)$ denote the evolution of D_0 under the flow. Let $V(t)$ denote the volume of D_t. Then

$$\frac{dV}{dt}|_{t=0} = \int_{D_0} \nabla \cdot f dx$$

where $\nabla \cdot f$ denotes the divergence of a vector field.

Examples: The pendulum equation and the Volterra-Lotka equation.

Liouville's Theorem. The flow generated by a time-independent Hamiltonian system is volume preserving,

Remark: Attraction by an equilibrium solution is impossible as the flow in phase space is volume preserving (Liouville).