Introduction to Autonomous Equations

CDS140A Lecturer: W.S. Koon

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1 Phase Space, Orbits

Consider *autonomous equations*

$$\dot{x} = f(x). \tag{1.0.1}$$

Example: Harmonic oscillator

$$\ddot{x} + x = 0.$$

Remarks:

- If $\phi(t)$ is a solution of equation (1.0.1), then $\phi(t - t_0)$ is also a solution.
- While the solutions $\phi(t)$ and $\phi(t - t_0)$ are different, they correspond to the same *orbital curve* in the *phase space* where the behavior of the variables $\{x_1, \ldots, x_n\}$ are described.

Orbits can be obtained as follow. Assume $f_1(x) \neq 0$. The chain rule gives a new set of $(n-1)$ differential equations with x_1 as its independent variable:

$$\frac{dx_i}{dx_1} = \frac{f_i(x)}{f_1(x)}, \quad i = 2, \ldots, n.$$

The solutions of this new system are orbits in the phase space. Clearly, at the point where $f_1(x) = 0$, another variable besides x_1 needs to act as the independent variables, and so on. Real problem arises at a point $\bar{x}$ where

$$f_i(\bar{x}) = 0 \quad \text{for all } i, \quad i = 1, 2, \ldots, n.$$

Then $\bar{x}$ is called a *critical point* or an *equilibrium point*. It corresponds to an equilibrium solution (or stationary solution).

Note: The existence and uniqueness theorem implies that orbits in phase space will not intersect.

2 Equilibrium Point and Linearization

- The first step to analyze a nonlinear system is to determine its equilibrium points and to describe the trajectory structures near these equilbrium points.
- For hyperbolic equilibrium points, their local behavior can be determined via its associated linearized system.
- For nonhyperbolic cases, other methods are needed.

Linearization: Let $\bar{x}$ be an equilibrium solution, lineariztion near $\bar{x}$ gives

$$\dot{u} = Df(\bar{x})u.$$

Example: Pendulum equation (saddle and center):

$$\ddot{x} + \sin x = 0.$$

Definition (Hyperbolic Equilibrium Point) An equilibrium point $\bar{x}$ is called a *hyperbolic equilibrium point* if none of the eigenvalues of the matrix $Df(\bar{x})$ have zero real part

Theorem 2.1 *(Hartman-Grobman) Let* 0 *be a hyperbolic equilibrium point of a nonlinear system* $\dot{x} = f(x)$ *with* $f \in C^1(E)$*. Then there exists a homeomorphism* H *which maps trajectories of the nonlinear system near the origin onto trajectories of its associated linearized system near the origin and preserve the parametrization by time, i.e.,*

$$H \circ \phi_t(x_0) = e^{At}H(x_0).$$

Example: Consider the system (center?)

$$\begin{aligned}\dot{x} &= -y + ax(x^2 + y^2)\\ \dot{y} &= x + ay(x^2 + y^2).\end{aligned}$$

3 Periodic Solutions

Definition: Suppose that $x = \phi(t)$ is a solution of the equation (1.0.1) and suppose that there exists a smallest positive number T such that $\phi(t + T) = \phi(t)$ for all $t \in \mathbb{R}$. Then $\phi(t)$ is called a periodic solution with period T.

Corollary 3.1 *A periodic solution of a autonomous equation corresponds to a closed orbit in the phase space and vise versa.*

Example: Consider the van der Pol equation:

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}, \qquad \mu > 0.$$

Remark: For non-autonomous equations, closed orbits do not necessarily correspond to periodic solutions as the translation property is not valid any more.

4 First Integral and Integral Manifolds

Recall: harmonic oscillator $\ddot{x} + x = 0$ has a first integral $F(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$.

Definition (Orbital Derivative): Consider the differential function $F : \mathbb{R}^n \to \mathbb{R}$ and the vector function $x : \mathbb{R} \to \mathbb{R}^n$. The *orbital derivative* L_t of F along x, parametrized by t is

$$L_t F = \frac{\partial F}{\partial x}\dot{x}.$$

Definition (Integral of Motions): The function $F(x)$ is called a first integral of the equation (1.0.1) if $L_t F = 0$ w.r.t. the solution $x(t)$.

Remarks:

- The first integral $F(x)$ is constant along a solution ("constant of motions").
- The level set of $F(x)$ contain orbits of the equation and is called *integral manifold* (useful in the study of the phase space).

Example: Consider the **Hamiltonian equations**

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \ \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

5 Morse Function

Morse lemma is useful for studying the local behavior of level sets near an equilibrium point.

Definition (Non-degenerate Critical Point): Consider $F : \mathbb{R}^n \to \mathbb{R}$, $x = a$ is called a non-degenerate critical point of F if

$$\frac{\partial F}{\partial x}(a) = 0, \qquad \left|\frac{\partial^2 F}{\partial x^2}(a)\right| \neq 0.$$

Definition (Morse Function): If $x = a$ is a non-degenerate critical point of F, then F is called a Morse function in a neighborhood of $x = a$.

Lemma 5.1 *(***Morse***) Suppose that F has a non-degenerate critical point at $x = 0$ with index k. Then there exists a diffeomorphism in a neighborhood of $x = 0$ which transform $F(x)$ to the form*

$$G(u) = G(0) - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 + \cdots + u_n^2.$$

Recall: Pendenlum equation

$$\ddot{x} + \sin x = 0.$$

Example: Consider the Volterra-Lotka equations

$$\begin{aligned}\dot{x} &= ax - bxy \\ \dot{y} &= bxy - cy\end{aligned}$$