# Existence-Uniqueness and Dependence on Initial Conditions

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## 1 Introduction

**Linear System.** Any linear system has a unique solution $x(t) = e^{At}x_0$ through each point $x_0 \in \mathbb{R}^n$ and the solution is defined for all $t \in \mathbb{R}$.

**Nonlinear Systems.** For nonlinear systems

$$\dot{x} = f(x)$$

where $f : E \to \mathbb{R}^n$ and $E$ is an open subset of $\mathbb{R}^n$, the situation is much more subtle.

- For example, the IVP $\dot{x} = \sqrt{x}, x(0) = 0$ has 2 different solutions.

- Even $f$ behaves nicely, $\dot{x} = x^2, x(0) = 1$, the solution become unbounded at some finite time.

We will show that

- under certain conditions of $f$, the nonlinear system has a unique solution through each point $x_0 \in E$ defined on a maximal interval of existence $(\alpha, \beta) \subset \mathbb{R}$;
- while it is generally impossible to solve the nonlinear system, a great deal of qualitative information about the local behavior of the solution can be determined via its associated linearized system $\dot{x} = Df(\bar{x})x$ where $\bar{x}$ is an equilibrium solution.

## 2 Existence and Uniqueness

**Definition:** Consider the function $f(t,x)$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $|t - t_0| \leq a, x \in E \subset \mathbb{R}^n$. $f(t,x)$ satisfies the Lipschitz condition (*Lipschitz continuous*) w.r.t. $x$ if in $[t_0 - a, t_0 + a] \times E$ we have

$$||f(t,x_1) - f(t,x_2)|| \leq L||x_1 - x_2||$$

with $x_1, x_2 \in E$ and $L$ a constant (Lipschitz constant).

**Remarks:**

- Lipschitz continuous in $x$ implies continuous in $x$.
- Continuous differentiability implies Lipschitz continuity.

**Recall:**

- $f \in C^1(E)$ if (i) $f$ is *differentiable* for all $x_0 \in E$, i.e., there exists a linear transformation $Df(x_0) \in L(\mathbb{R}^n)$ such that

  $$\lim_{|h| \to 0} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)h|}{|h|} = 0$$

  and (ii) $Df : E \to L(\mathbb{R}^n)$ is continuous.
- $\mathbf{f} \in C^1(E)$ iff $\frac{\partial f_i}{\partial x_j}$ exist and are continuous on $E$. $Df = \left[\frac{\partial f_i}{\partial x_j}\right]$.

**Theorem 2.1** *Consider the IVP*

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0$$

*with* $x \in E \subset \mathbb{R}^n, |t - t_0| \leq a; E = \{x | ||x - x_0|| \leq d\}$. *If*

- $f(t,x)$ *is continuous in* $G = [t_0 - a, t_0 + a] \times E$ *and*
- $f(t,x)$ *is Lipschitz continuous in* $x$,

*then the IVP has one and only one solution for* $|t - t_0| \leq \min(a, d/M)$ *with* $M = sup_G ||f||$.

**Theorem 2.2** *Let* $x_0 \in E \subset \mathbb{R}^n$. *If* $f \in C^1(E)$, *then there exists* $b > 0$ *such that the IVP has a unique solution* $x(t)$ *on* $[-b, b]$.

**Proof:** Based on Picard's method of successive approximations.

1. Notice that $x(t)$ is a solution of the IVP iff it is a continuous and satisfies the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))ds.$$

2. The successive approximations to the solution are defined by

$$\begin{aligned} u_0(t) &= x_0 \\ u_{k+1}(t) &= x_0 + \int_0^t f(u_k(s))ds \end{aligned}$$

3. For example, consider $\dot{x} = x, x(0) = 1$.

4. To show that $u_k$ converge to a solution. Need to recall:
   - $C([-a, a])$ (set of continuous function on $[-a.a]$) is a complete normed linear space: every Cauchy sequence converges.
   - If $f \in C^1(E)$, then $f$ is locally Lipschitz on $E$.

5. The key step is to show that $\{u_k\}$ is a Cauchy sequence of continuous functions.

6. Argument for existence:

7. Argument for uniqueness:

**Remarks:** The solution of the IVP will be written as $x(t)$, $x(t; x_0)$ or $x(t; t_0, x_0)$.

- The theorem guarantees the existence of the solution in a neighborhood of $t = t_0$, the size of which depends on the supnorm $M$ of $f(t, x)$.
- One often can continue the solution outside this neighborhood.

**Theorem 2.3** *If $x \in C^1(M)$ with $M$ compact, there the system has solution curves defined for all $t \in \mathbb{R}^n$.*

# 3 Dependence on Initial Conditions

**Theorem 3.1** *Under the same hypothesis as theorem (2.1), If $||\eta|| \leq \epsilon$ then we have*

$$||x_0(t) - x_\epsilon(t)|| \leq \epsilon e^{Lt} \quad \text{on} \quad I$$

*where $x_0(t), x_\epsilon(t)$ are the solutions to the IVPs $\dot{x} = f(t, x), x_0(0) = a$ and $\dot{x} = f(t, x), x_\epsilon(0) = a + \eta$ on interval $I$ respectively.*

**Theorem 3.2** *(**Gronwall**) Assume that for $t_0 \leq t \leq t_0 + a$,*

$$\phi(t) \leq \delta_1 \int_{t_0}^{t} \psi(s)\phi(s)ds + \delta_3$$

*where $\phi(t) \geq 0, \psi(t) \geq 0, \delta_1 > 0, \delta_3 > 0$. Then*

$$\phi(t) \leq \delta_3 e^{\delta_1 \int_{t_o}^{t} \psi(s)ds}.$$

**Theorem 3.3** *Under the same hypothesis as above, if*

$$\phi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \phi(s)ds + \delta_3$$

*then*

$$\phi(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) e^{\delta_1(t-t_0)} - \frac{\delta_2}{\delta_1}.$$