Lecture 5: Potential Games and the Inefficiency of Equilibria

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Outline

- Potential games
  - Review on strategic games
  - Potential games (atomic and nonatomic)

- Inefficiency of equilibria
  - The price of anarchy and selfish routing
  - Resource allocation
  - Network design games and the price of stability
Strategic game

- **Def:** A game in strategic form is a triple

\[ G = \{ N, S_{i \in N}, u_{i \in N} \} \]

- \( N \) is the set of players (agents)
- \( S_i \) is the player \( i \) strategy space
- \( u_i : S \rightarrow R \) is the player \( i \) payoff function

- **Notations**
  - \( S = S_1 \times S_2 \times \cdots \times S_N \): the set of all profiles of player strategies
  - \( s = (s_1, s_2, \ldots, s_N) \): profile of strategies
  - \( s_{-i} = (s_1, s_2, \ldots, s_{i-1}, \ldots, s_{i+1}, \ldots, s_N) \): the profile of strategies other than player \( i \)
Implicitly assume that players have preferences over different outcomes, which can be captured by assigning payoffs to the outcomes.

The basic model of rationality is that of a payoff maximizer (or cost minimizer).
Example: finite game

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Example: Continuous strategy game

- **Cournot competition**
  - Two players: firm 1 and firm 2
  - Strategy $s_i \in [0, \infty]$: the amount of widget that firm $i$ produces
  - The payoff for each firm is the net revenue
    \[
    u_i(s_1, s_2) = s_i p(s_1 + s_2) - c_i s_i
    \]
    where $p$ is the price, $c_i$ is the unit cost for firm $i$
Nash equilibrium

- Def: A strategy profile \( s^* \) is a Nash equilibrium, if for all \( i \),
  \[
  u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \quad \text{for all } s_i \in S_i
  \]

- For any \( s_{-i} \in S_{-i} \), define best response function
  \[
  B_i(s_{-i}) = \{ s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i \}.
  \]

Then a strategy profile \( s^* \) is a Nash equilibrium
iff \( s^*_i \in B_i(s^*_{-i}) \).
Examples

- **Battle of the Sexes**

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Two Nash equilibria (Ballet, Ballet) and (Soccer, Soccer)
Cournot Competition

- Suppose a price function \( p(s_1 + s_2) = \max\{0, 1 - (s_1 + s_2)\} \)
- Suppose cost \( 0 \leq c_1 = c_2 = c \leq 1 \)
- Then, the best response function

\[
B_1(s_2) = \frac{(1 - s_2 - c)}{2}, \quad B_2(s_1) = \frac{(1 - s_1 - c)}{2}
\]

- Nash equilibrium satisfies \( s_1 = B_1(s_2), s_2 = B_2(s_1) \), i.e., 
  \[
  s_1 = \frac{(1 - c)}{3}, \quad s_2 = \frac{(1 - c)}{3}
  \]
Potential games (atomic)

- **Def:** A function $\Phi : S \rightarrow R$ is a potential function for game $G$ if for $\forall i, \forall s_{-i} \in S_{-i}, \forall s_i, \overline{s_i} \in S_i$,

$$u_i(s_i, s_{-i}) - u_i(\overline{s_i}, s_{-i}) = \Phi(s_i, s_{-i}) - \Phi(\overline{s_i}, s_{-i}).$$

When $\Phi$ exists, the game is called a potential game.

- **Def:** A function $\Phi : S \rightarrow R$ is an ordinal potential function for game $G$ if for $\forall i, \forall s_{-i} \in S_{-i}, \forall s_i, \overline{s_i} \in S_i$,

$$u_i(s_i, s_{-i}) - u_i(\overline{s_i}, s_{-i}) > 0 \iff \Phi(s_i, s_{-i}) - \Phi(\overline{s_i}, s_{-i}) > 0.$$

When $\Phi$ exists, the game is called an ordinal potential game.
Example

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<th>Game</th>
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**Equilibrium**

- $s^*$ is a pure strategy Nash equilibrium for ordinal potential game $G$, iff

\[ \Phi(s^*_i, s^*_i) \geq \Phi(s^*_i, s^*_{-i}), \forall i, \forall s_i \in S_i. \]

- **Proof:** If $\Phi$ is a potential function

\[ u_i(s^*_i, s^*_i) - u_i(s_i, s^*_i) = 0 \iff \Phi(s^*_i, s^*_i) - \Phi(s_i, s^*_i) = 0. \]

$s^*$ is a pure strategy Nash equilibrium, iff

\[ u_i(s^*_i, s^*_i) \geq u_i(s_i, s^*_i) \iff \Phi(s^*_i, s^*_i) \geq \Phi(s_i, s^*_i). \]
If $\Phi$ has a maximum at $s^*$, then $s^*$ is a pure strategy Nash equilibrium of the ordinal game.

Every finite ordinal potential game has a pure strategy Nash equilibrium.

Continuous ordinal potential game has a pure strategy Nash equilibrium if the strategy space is compact and potential is continuous.
**Congestion games**

- **Def**: A congestion model $\{N, M, S_{i\in N}, c_{j\in M}\}$ is defined as follows
  - $N$ is the set of players
  - $M$ is the set of facilities or resources
  - $S_i$ is the sets of the resources that player $i$ can use
  - $c_j(k_j)$ is the cost to users who use the resource $j$ when $k_j$ users are using it

- **Def**: A congestion game associated with a congestion model is a game $\{N, S_{i\in N}, c_{i\in N}\}$ with cost $c_i(s) = \sum_{j\in S_i} c_j(k_j)$
Every congestion game is a potential game, with potential

$$\Phi(s) = \sum_{j \in \cup s_i} \sum_{k=1}^{k_j} c_j(k).$$

Congestion games have many applications
- Network design
Potential games (nonatomic)

- Nonatomic game: the user number is infinite
  - $N$ classes of “infinitesimal” players
  - $r_i$ the “mass” of class $i$ players
  - $f(i, s_i)$ the fraction of class $i$ players that choose strategy $s_i$
  - $u_i(s_i; f)$ the payoff for a player of class $i$ with $s_i$

- **Def:** $f^*$ is an equilibrium if for all $\forall i, \forall s_i, \bar{s}_i \in S_i$,
  $$f^*(i, s_i) > 0 \Rightarrow u_i(s_i; f^*) \geq u_i(\bar{s}_i; f^*).$$

- **Def:** A nonatomic game is a potential game if there exists potential function $\Phi(f)$ such that
  $$u_i(s_i; f) = \frac{\partial \Phi(f)}{\partial f(i, s_i)}.$$
Example: selfish routing

- Consider a multicommodity flow network \((V, E)\)
  - \(N\) source-destination pairs (commodities)
  - Each commodity \(i\) has a total rate \(r_i\), and can use a set \(P_i\) of paths
  - The aggregate traffic among link \(e\)
    \[ f_e = \sum_{i, s_i \in P_i} f(i, s_i) \]
  - \(c_e(f_e)\) link \(e\) cost, a nonnegative, continuous non-decreasing function of traffic \(f_e\)
  - The cost \(c_i(s_i; f) = \sum_{e \in P_i} c_e(f_e)\)
- Wardrop equilibrium: the costs of all the paths actually used are equal, and less than those which would be experienced by a single user on any unused path.
- $\{V, E; r, c\}$ is a potential game with potential
  \[
  \Phi(f) = \sum_e \int_0^{f_e} c_e(x)\,dx.
  \]
Inefficiency of equilibria

- Equilibria of strategic games are typically inefficient
- Example: Prisoner's Dilemma

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Pigou’s example

One commodity with rate 1

- A unique Wardrop equilibrium, with all traffic routed on the lower edge
- A better flow: route half of the traffic on each of the two edges
Quantify the inefficiency

- Price of anarchy: quantify inefficiency with respect to some objective function

\[
\text{price of anarchy} = \frac{\text{obj fn value of the worst equilibrium}}{\text{optimal obj fn value}}
\]

- \( \leq 1 \) for maximization; \( \geq 1 \) for minimization

- Interested in situations in which we can bound the price of anarchy
Selfish Routing

- At Wardrop equilibrium
  \[ f(i, s_i)(c_i(s_i; f) - c_i(f)) = 0 \]
  \[ c_i(s_i; f) - c_i(f) \geq 0 \]
  \[ \sum_{s_i} f(i, s_i) = r_i \]
  \[ c_i(f) = \min_{s_i} c_i(s_i, f) \]

- The above is the KKT optimality condition for
  \[
  \min_f \quad \Phi(f) = \sum_e \int_0^{f_e} c_e(x) \, dx \\
  \text{s.t.} \quad \sum_{s_i} f(i, s_i) = r_i \\
  \sum_{i, s_i : e \in s_i} f(i, s_i) = f_e
  \]
A flow $f$ for $\{V, E; r, c\}$ is a Wardrop equilibrium if and only if it is a global minimum of the potential function

$$\Phi(f) = \sum_e \int_0^{f_e} c_e(x) \, dx.$$ 

Define the objective function, i.e., the cost of flow as

$$C(f) = \sum_{i,s_i} f(i,s_i)c_i(s_i,f) = \sum_e f_e c_e(f_e)$$

Def: An optimal flow $f^*$ for $\{V, E; r, c\}$ is the flow that minimizes $C(f)$. 

The price of anarchy

- The price of anarchy is
  \[ \rho = \frac{C(f)}{C(f^*)}. \]

- Pigou’s example \( \rho = 4/3 \)

- Suppose that \( x \cdot c_e(x) \leq \gamma \cdot \int_0^x c_e(y)\,dy \), then \( \rho \leq \gamma \).

- Pigou’s example with degree-d polynomial cost
  - \( x \cdot c_e(x) \leq (d+1) \cdot \int_0^x c_e(y)\,dy \)
  - \( \rho \leq d + 1 \)

- Tight bounds are 4/3 and \( d / \ln d \) respectively.
Consider a simple network: the sources (users) share a link and the network (link) manager wants to allocate link rate such that

\[
\text{System:} \quad \max_x \sum_s U_s(x_s) \\
\text{s.t.} \quad \sum_s x_s \leq c
\]

Utility functions are not known to the link manager.
Market-clearing mechanism

- Each user $s$ submits a bid (or willingness to pay) $w_s$.
- The manager seeks to allocate the entire link capacity, and sets a price $p$ such that

$$\sum_s \frac{w_s}{p} = c$$

- As if the user has a demand function

$$D(p, w_s) = \frac{w_s}{p}$$

- The link manager chooses a price to clear the market

$$\sum_s D(p, w_s) = c$$
Price taking users and competitive equilibrium

- The user is a price taker: does not anticipate the effect of his payment on the price
- It is rational for the user to maximize the following payoff (Kelly '98)

\[ u_s(p, w_s) = U_s\left(\frac{w_s}{p}\right) - w_s \]

- A pair \((p, w)\) is a competitive equilibrium if

\[ u_s(p, w_s) \geq u_s(p, \bar{w}_s) \text{ for any } \bar{w}_s \geq 0 \]

\[ p = \left(\sum_s w_s\right) / c \]
Theorem (Kelly ’98): there exist a unique competitive equilibrium \((p, w)\) such that \(x = w/p\) solves the problem \(\text{System}\).

Proof: consider the Lagrangian
\[
D(p, x) = \sum_s U(x_s) - p(\sum_s x_s - c)
\]

At primal-dual optimal
\[
U'_s(x_s) = p, \text{ if } x_s > 0 \\
U'_s(x_s) \leq p, \text{ if } x_s = 0 \\
p \geq 0 \\
p(\sum_s x_s - c) = 0
\]
Since $c > 0$, at least one $x_s$ is positive. So, $p > 0$.

Thus, $\sum_{s} x_s = c$.

Let $w = px$, then $(p, w)$ is a competitive equilibrium and $x = w/p$ solves the problem $System$.

In this case, the uniqueness of $x$ follows from the uniqueness of $p$. 
Price anticipating users and Nash equilibrium

- Price anticipating users realizes that the price is set according to \( p = (\sum s w_s) / c \), and will adjust their bids accordingly.

- This makes the model a game, where user payoff is (Johari ’04)

\[
    u_s(w_s, w_{-w}) = \begin{cases} 
        U_s\left(\frac{w_s}{\sum_s w_s} c\right) - w_s, & \text{if } w_s > 0 \\
        U_s(0), & \text{if } w_s = 0 
    \end{cases}
\]

- Consider Nash equilibrium \( w \) such that

\[
    u_s(w_s, w_{-s}) \geq u_s(w_{\bar{s}}, w_{-s}), \text{ for all } \bar{w} \geq 0, \text{ for all } s
\]
Theorem (Hajek, et al): there exists a unique Nash equilibrium \( w \geq 0 \). Moreover, the rates are unique solution of the following problem

\[
\text{Game: } \max_x \sum_s \hat{U}_s(x_s) \\
\text{s.t. } \sum_s x_s \leq c
\]

where

\[
\hat{U}_s(x_s) = (1 - \frac{x_s}{c})U_s(x_s) + \frac{x_s}{c} \left( \frac{1}{x_s} \int_0^{x_s} U_s(z) \, dz \right).
\]
Proof:

- If $w$ is a Nash equilibrium, at least two players have nonzero bids.
- Then $u_s(w_s, w_{-w})$ is strictly concave and continuously differentiable in $w_s$.
- Then, at equilibrium

$$U'_s \left( \frac{w_s}{\sum_t w_t} c \right) \left( 1 - \frac{w_s}{\sum_t w_t} \right) = \frac{\sum_t w_t}{c}, \text{ if } w_s > 0$$

$$U'_s(0) \leq \frac{\sum_t w_t}{c}, \text{ if } w_s = 0$$

- The above condition is also sufficient.
The problem \textit{Game} has a unique optimal $x$. Moreover, there exist a $p$ such that

\[ U'_s(x_s)(1 - \frac{x_s}{c}) = p, \text{ if } x_s > 0 \]

\[ U'_s(x_s) \leq p, \text{ if } x_s = 0 \]

$p \geq 0$

$p\left(\sum_s x_s - c\right) = 0$

Let $x = \frac{w}{p}$ and $p = \sum_t \frac{w_t}{c}$. Then $(x, p)$ satisfies the above optimality condition.
The price of anarchy

- Assume $U_s(0) \geq 0$, we have
  $$\frac{1}{x_s} \int_0^{x_s} U_s(z)dz \leq U_s(x_s)$$

- Then $\hat{U}_s(x_s) \leq U_s(x_s)$

- Since $U_s(z) \geq \frac{z}{x_s} U_s(x_s) + (1 - \frac{z}{x_s})U_s(0)$, $0 \leq z \leq x_s$, we have
  $$\int_0^{x_s} U_s(z)dz \geq \frac{x_s}{2} U_s(x_s)$$

- Then, $\hat{U}_s(x_s) \geq \frac{1}{2} U_s(x_s)$
Let \( x^* \) and \( x \) are the optima of problems \textit{System} and \textit{Game}, we have

\[
\frac{1}{2} \sum_s U_s(x^*_s) \leq \sum_s \hat{U}_s(x^*_s) \leq \sum_s \hat{U}_s(x_s) \leq \sum_s U_s(x_s)
\]

- The price of anarchy \( \rho \geq 1/2 \)
Tight bound

- Define the JT bound $\beta$ by
  \[
  \beta = \inf_{U} \inf_{c} \inf_{0 \leq x, x^* \leq c} \frac{U(x) + \hat{U}'(x)(x^* - x)}{U(x^*)}
  \]

- For any $\varepsilon > 0$, there is a resource allocation game with the price of anarchy at most $\beta + \varepsilon$.
  - Proof: first note that we can assume $x < x^*$ & $c = x^*$.
  - Define a game with $U_1(x_1) = U(x_1)$
    \[
    U_s(x_s) = \hat{U}'(x) \cdot x_s, \ s \geq 2
    \]
  - At optimal, the efficiency is $U_1(c) = U(x^*)$
  - At equilibrium $\hat{U}'(x_1) = \hat{U}'(x) = \hat{U}'(x) = \hat{U}'(x)(1 - x_s / C)$
Then, $x_1 \to x$ as the player number goes to infinity.

Thus, the efficiency at equilibrium approaching

$$\hat{U}_1(x) + \hat{U}'(x)(C - x) = \hat{U}(x) + \hat{U}'(x)(x^* - x).$$

In every resource allocation game, the price of anarchy is at least $\beta$.

Proof: let $x^*$ and $x$ are the optimal and equilibrium

$$\sum_s U_s(x^*_s) \leq \sum_s \frac{i}{\beta} (U_s(x_s) + \hat{U}_s'(x_s)(x^*_s - x_s)) \leq \frac{1}{\beta} \sum_s U_s(x_s)$$
The bound $\beta = 3/4$.

Proof: setting $U(x) = x$ & $x = 1/2$ & $c = x^* = 1$ shows the bound is at most $3/4$.

Assume $x < x^* = c$, we have

$$U(x) + \hat{U}'(x)(x^* - x) = U(x) + (1 - x / x^*)U'(x)(x^* - x)$$
$$\geq U(x) + (1 - x / x^*)(U(x^*) - U(x))$$
$$= (x / x^*)U(x) + (1 - x / x^*)U(x^*)$$
$$\geq (x / x^*)^2 U(x) + (1 - x / x^*)U(x^*)$$
$$\geq \frac{3}{4} U(x^*).$$
Consider a network $(V,E)$ with a nonnegative cost $c_e$ for each edge $e \in E$.

- $N$ source-destination pairs (players)
- Each player $i$ can choose a path $s_i \in P_i$
- The total cost is $c(s) = \sum_{e \in \bigcup s_i} c_e$

Let $n_e$ denote the number of players whose paths are using edge $e$. Each of those players pays a share $\pi_e = c_e / n_e$ of the cost.

The cost for each player $i$ is $c_i(s_i; s_{-i}) = \sum_{e \in s_i} c_e / n_e$
\( \{V, E; c\} \) is a potential game with potential function
\[
\Phi(s) = \sum_{e} \sum_{j=1}^{n_e} \frac{c_e}{j}.
\]

Every network design game has at least one Nash equilibrium.
$k$ players and $a > 0$ arbitrarily small

- Two Nash equilibria: all chooses the upper edges, or all choose the lower edge
Price of stability

- **Price of stability**
  
  \[
  \text{price of stability} = \frac{\text{obj fn value of the best equilibrium}}{\text{optimal obj fn value}}
  \]

- **Since** \( C(s) \leq \Phi(s) \leq (1 + 1/2 + \cdots + 1/k)C(s) \), the **price of stability** is at most \( 1 + 1/2 + \cdots + 1/k \).