Lecture 3: Duality Model of TCP/AQM and Market Mechanism for Resource Allocation

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Agenda

- The duality model of TCP/AQM
  - Utility maximization and dual decomposition
  - An introduction to TCP congestion control
  - A general dynamics model of TCP/AQM
  - The duality model of TCP/AQM

- Market mechanism for resource allocation
  - Market-clearing mechanism
  - Price-taking and competitive equilibrium
  - Price-anticipating and Nash equilibrium
Network model

- A network modeled as a set \( L \) of links with finite capacities \( c = \{c_l, l \in L\} \)
- Shared by a set \( S \) of sources
- Each source \( s \) uses a subset \( L_s \subseteq L \) of links, which defines a routing matrix

\[
R_{ls} = \begin{cases} 
1 & \text{if } l \in L_s \\
0 & \text{otherwise}
\end{cases}
\]

\[
R = \begin{pmatrix} 
1 & 1 & 0 \\
1 & 0 & 1 
\end{pmatrix}
\]
Resource allocation problem

- Each source $s$ attains a utility $U_s(x_s)$ when transmitting at rate $x_s$
- Utility maximization (Kelly '98)

$$\max_x \sum_s U_s(x_s)$$

s.t. $Rx \leq c$

$$Rx = \begin{pmatrix} 110 \\ 101 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
For elastic traffic, $U_s(\cdot)$ is assumed to be continuously differentiable, increasing and strictly concave.

Ensure some kind of fairness.

Polynomial-solvable, if all the utility and constraint information is provided. But impractical in real networks.

Have to seek decomposition to obtain distributed algorithm.

$$\max_x \sum_s U_s(x_s)$$

$$s.t. \quad Rx \leq c$$
Consider the dual problem (Low ’99)

\[
\min_{p \geq 0} D(p) := \max_x \sum_s U_s(x_s) - p^T (Rx - c) = \max_x \sum_s (U_s(x_s) - x_s \sum_c R_{ls} p_l) + p^T c
\]

- Congestion control: given end-to-end price \( q_s = \sum_l R_{ls} p_l \)
  \[x_s(t) = U_s^{-1}(q_s(t))\]

- Price update: given aggregate source rate \( y_i = \sum_s R_{is} x_s \)
  \[p_l(t + 1) = [p_l(t) + \gamma (y_i(t) - c_i)]^+\]

- Prices can be updated and fed back to sources implicitly
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TCP/IP protocol stack

- Does the part of task specific to the particular applications
- Provide reliable end-to-end transmission, congestion control
- Provide end-to-end path between two end nodes (routing)
- Provide reliable point-to-point transmission, channel access
- Provide a link for transmitting bits between two nodes
Congestion control

- **Effect of congestion**
  - Packet loss, retransmission, reduced throughput even congestion collapse
  - Internet has its first congestion collapse in Oct. 1986

- **Congestion control**
  - Achieve high utilization
  - Avoid congestion
  - Fair bandwidth sharing
Window-based flow control

- \( \approx W \) packets per RTT (round trip time)
- Lost packet detected by missing ACK
- Source rate \( \approx W/RTT \)
TCP congestion control

- Source calculates cwnd from indication of network congestion
  - cwnd: congestion window size

- Congestion indications
  - Packet losses
  - Delay
  - Packet marks

- Algorithms to calculate cwnd
  - Tahoe, Reno, Vegas, …
  - DropTail, RED, REM, …
TCP Reno (Jacobson '90)

SS: Slow Start
CA: Congestion Avoidance
Slow start

- Start with $cwnd = 1$ (slow start)
- On each successful ACK increment $cwnd$
  
  $cwnd \leftarrow cnwd + 1$

- **Exponential growth of $cwnd$**
  
  each RTT: $cwnd \leftarrow 2 \times cwnd$

- Enter **CA** when $cwnd \geq ssthresh$
Congestion Avoidance

- Starts when \( cwnd \geq ssthresh \)
- On each successful ACK:
  \[ cwnd \leftarrow cwnd + \frac{1}{cwnd} \]
- Linear growth of \( cwnd \)
  - each RTT: \( cwnd \leftarrow cwnd + 1 \)
Packet loss

- **Assumption**: loss indicates congestion
- Packet loss detected by
  - Retransmission TimeOuts (RTO timer)
  - Duplicate ACKs (at least 3)
- Upon packet loss
  - Retransmit lost packet
  - Set $ssthresh \leftarrow cwnd/2$
  - Set $cwnd \leftarrow ssthresh$
  - Enter CA
- Called fast retransmission/fast recovery
Summary: Reno

- Basic ideas
  - Gently probe network for spare capacity
  - Drastically reduce rate on congestion

```plaintext
for every ACK {
    if (W < ssthresh) then W++  (SS)
    else W += 1/W       (CA)
}
for every loss {
    ssthresh = W/2
    W = ssthresh
}
```
TCP Vegas (Brakmo & Peterson '94)

Congestion measure: end-to-end queueing delay

For every RTT
\[ \{ \begin{align*} & \text{if } W_{\text{RTT}_\text{min}} - W_{\text{RTT}} < \alpha \text{ then } W++ \\ & \text{if } W_{\text{RTT}_\text{min}} - W_{\text{RTT}} > \alpha \text{ then } W-- \end{align*} \} \]

For every loss
\[ W := W/2 \]
Link algorithms

- DropTail: drop coming packets when buffer is full
- RED: warn sources of incipient congestion by probabilistically marking/dropping packets
  - Probabilistically drop packets
  - Probabilistically mark packets
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TCP & AQM

Two components

- A source algorithm: adjust the sending rate based on congestion
  - Implemented in TCP (Transmission Control Protocol)

- A link algorithm: update a congestion measure and send it back to the sources
  - Congestion measure: loss probability and delay
  - In form of loss or delay
  - Carried out by AQM (Active Queuing Management)
Dynamic model of TCP/AQM

- **Notation**
  - $x_s(t)$: source rate at time $t$
  - $y_l(t) = \sum R_{ls} x_s(t)$: aggregate source rate at link $l$
  - $p_l(t)$: link congestion measure
  - $q_s(t) = \sum R_{ls} p_l(t)$: end-to-end congestion measure of source $S$

- **Source** $S$ can observe only $x_s(t)$ and $q_s(t)$
- **Link** $l$ can observe only $p_l(t)$ and $y_l(t)$
Dynamical model

\[ x_s(t + 1) = F_s(x_s(t), q_s(t)) \]
\[ p_l(t + 1) = G_l(p_l(t), y_l(t)) \]

The exact forms of \( F_s \) and \( G_l \) are determined by the specific TCP/AQM protocol.
Example: Vegas

\[ D_s = d_s + q_s \]

For every RTT

\[
\left\{ \begin{array}{ll}
\text{if } & W_{\text{RTT}_{\text{min}}} - W_{\text{RTT}} < \alpha \\
& \text{then } W++ \\
\text{if } & W_{\text{RTT}_{\text{min}}} - W_{\text{RTT}} > \alpha \\
& \text{then } W-- \\
\end{array} \right. 
\]

For every loss

\[ W := W/2 \]

\[
F: \quad x_s(t+1) = \begin{cases} 
  x_s(t) + \frac{1}{D_s^2} & \text{if } w_s(t) - ds x_s(t) < \alpha_s d_s \\
  x_s(t) - \frac{1}{D_s^2} & \text{if } w_s(t) - ds x_s(t) > \alpha_s d_s \\
  x_s(t) & \text{else}
\end{cases}
\]

\[
G: \quad p_l(t+1) = [p_l(t) + y_l(t)/c_l - 1]^+ 
\]
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Duality model of TCP/AQM

- Denote by \((x^*, p^*)\) the equilibrium of the system
  \[
  x_s(t + 1) = F_s(x_s(t), q_s(t)) \\
  p_l(t + 1) = G_l(p_l(t), y_l(t))
  \]

- The fixed point equation \(x_s^* = F_s(x_s^*, q_s^*)\) implicitly define a relation \(q_s^* = f_s(x_s^*) > 0\)
  - \(F_s\) is continuously differentiable, and \(\partial F_s/\partial q_s \neq 0\)

- Define a utility function for each source
  \[
  U_s(x_s) = \int f_s(x_s)dx_s, \ x_s > 0
  \]
  - Usually continuous, increasing and strictly concave
  - Only determined by tcp algorithms
**Example: Vegas**

\[ F: \quad x_s(t + 1) = \begin{cases} 
  x_s(t) + \frac{1}{D_s^2} & \text{if } w_s(t) - d_s x_s(t) < \alpha_s d_s \\
  x_s(t) - \frac{1}{D_s^2} & \text{if } w_s(t) - d_s x_s(t) > \alpha_s d_s \\
  x_s(t) & \text{else}
\end{cases} \]

- **At equilibrium**
  \[ w_s^* - d_s x_s^* = \alpha_s d_s \Rightarrow x_s^* (D_s^* - d_s) = \alpha_s d_s \]
  \[ \Rightarrow x_s^* q_s^* = \alpha_s d_s \Rightarrow q_s^* = \alpha_s d_s / x_s^* \]

- **Utility function**
  \[ U_s(x_s) = \alpha_s d_s \log x_s \]
Define utility maximization

\[ \max_x \sum_s U_s(x_s) \]

\[ \text{s.t. } Rx \leq c \]

Dual problem

\[ \min_{p \geq 0} D = \max_x \sum_s (U_s(x_s) - x_s \sum_c R_{ls} p_l) + p^T c \]

Interpret source rate \( x \) as primal variable and the congestion price \( p \) as dual variable
The equilibrium \((x^*, p^*)\) solves the primal and dual, if it satisfies KKT condition

\[ y^*_l \leq c_l \]
\[ p^*_l (y^*_l - c_l) = 0 \]
\[ p^*_l \geq 0 \]
\[ U'_s(x^*_s) - q^*_s = 0 \]

- The complementary slackness condition is satisfied by any AQM that stabilize the queues
- AQM should match input rate to capacity to maximize utilization at every bottleneck link
Remarks

- Reverse engineering: the network as the optimization solver, and different TCP/AQM protocols as distributed primal-dual algorithms to solve the utility maximization and its dual

- Forward engineering: guide new congestion control design
  - By carefully choosing utility function
  - By proposing better convergent algorithm
Can extend to provide a mathematical theory for network architecture and a general approach to cross-layer design

- Minimize response time (web layout)...
- Maximize utility (TCP/AQM)
- Minimize path costs (IP)
- Throughput-maximal scheduling, ...
- Minimize SIR, max capacities, ...

Will discuss this in 4th lecture
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A simple resource allocation problem

Consider a simple version: the sources (users) share a link and the network (link) manager wants to allocate link rate such that

System:
\[
\max_x \sum_s U_s(x_s)
\]
\[
s.t. \sum_s x_s \leq c
\]

Utility functions are not known to the link manager
Market-clearing mechanism

- Each user $s$ submits a bid (or willingness to pay) $w_s$
- The manager seeks to allocate the entire link capacity, and sets a price $p$ such that
  \[ \sum_s \frac{w_s}{p} = c \]

- As if the user has a demand function
  \[ D(p, w_s) = \frac{w_s}{p} \]
- The link manager chooses a price to clear the market
  \[ \sum_s D(p, w_s) = c \]
Price taking users and competitive equilibrium

- The user is a price taker: does not anticipate the effect of his payment on the price.

- It is rational for the user to maximize the following payoff (Kelly ’98)

\[ u_s(p, w_s) = U_s\left(\frac{w_s}{p}\right) - w_s \]

- A pair \((p, w)\) is a competitive equilibrium if

\[ u_s(p, w_s) \geq u_s(p, \bar{w}_s) \quad \text{for any} \quad \bar{w}_s \geq 0 \]

\[ p = \left(\sum_s w_s\right) / c \]
Theorem (Kelly ’98): there exist a unique competitive equilibrium \((p, w)\) such that \(x = w / p\) solves the problem System.

Proof: consider the Lagrangian

\[
D(p, x) = \sum_s U(x_s) - p(\sum_s x_s - c)
\]

At primal-dual optimal

- \(U'_s(x_s) = p, \text{ if } x_s > 0\)
- \(U'_s(x_s) \leq p, \text{ if } x_s = 0\)
- \(p \geq 0\)
- \(p(\sum_s x_s - c) = 0\)
Since $c > 0$, at least one $x_s$ is positive. So, $p > 0$.

Thus, $\sum x_s = c$.

Let $w = px$, then $(p, w)$ is a competitive equilibrium and $x = w/p$ solves the problem System.

In this case, the uniqueness of $p$ follows from the uniqueness of $x$. 
Price anticipating users and Nash equilibrium

- Price anticipating users realizes that the price is set according to \( p = \left( \sum_{s} w_s \right) / c \), and will adjust their bids accordingly.

- This makes the model a game, where user payoff is (Johari '04)

\[
\begin{align*}
  u_s(w_s, w_{-w}) &= \begin{cases} 
    U_s \left( \frac{w_s}{\sum_s w_s} c \right) - w_s, & \text{if } w_s > 0 \\
    U_s(0), & \text{if } w_s = 0
  \end{cases}
\end{align*}
\]

- Consider Nash equilibrium \( w \) such that

\[
  u_s(w_s, w_{-s}) \geq u_s(\bar{w}_s, w_{-s}), \text{ for all } \bar{w} \geq 0, \text{ for all } s
\]
Theorem (Hajek, et al): there exists a unique Nash equilibrium \( w \geq 0 \). Moreover, the rates \( x_s = \frac{w_s}{\sum_s w_s} c \) are unique solution of the following problem

Game:

\[
\max_x \sum_s \hat{U}_s(x_s)
\]

s.t. \( \sum_s x_s \leq c \)

where

\[
\hat{U}_s(x_s) = \left(1 - \frac{x_s}{c} \right) U_s(x_s) + \frac{x_s}{c} \left( \frac{1}{x_s} \int_0^{x_s} U_s(z)dz \right).
\]
Remarks

- Will provide the proof in the 5th lecture
  - Closely relate to potential games
- Will characterize the efficiency loss of the Nash equilibrium in that lecture.
  - Selfish behavior can lead to inefficient outcomes.
    - E.g., Prisoner's Dilemma game
      \[
      \begin{array}{cc}
      D & C \\
      D & 3,3 & 0,4 \\
      C & 4,0 & 1,1 \\
      \end{array}
      \]
  - The price of anarchy
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