The Hunter–Saxton Equation: A Geometric Approach

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Abstract

The Hunter–Saxton equation is the Euler equation for the geodesic flow on the quotient space of the infinite-dimensional group of orientation preserving diffeomorphisms of the unit circle modulo the subgroup of rigid rotations equipped with a right-invariant metric. We establish several properties of this quotient space: it has constant sectional curvature equal to 1, the Riemannian exponential map provides global normal coordinates, and there exists a unique length-minimizing geodesic joining any two points of the manifold. Moreover, we give explicit formulas for the Jacobi fields, we prove that the diameter of the space is $\frac{\pi}{2}$, and give exact estimates for how fast the geodesics spread apart. At the end, these results are given a simple geometric and intuitive understanding when an isometry from the quotient space to an open subset of an L^2-sphere is constructed.