Oscillating Chemical Reaction

- Consider (x, y are concentrations of I^-, ClO_2^-)

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad \dot{y} = bx\left(1 - \frac{y}{1+x^2}\right).$$

- Construct trapping region for Poincaré-Bendixson theorem by sketching nullclines

$$y = \frac{(a-x)(1+x^2)}{4x}; \qquad x = 0, \qquad y = 1 + x^2.$$

- Dashed box is trapping region.

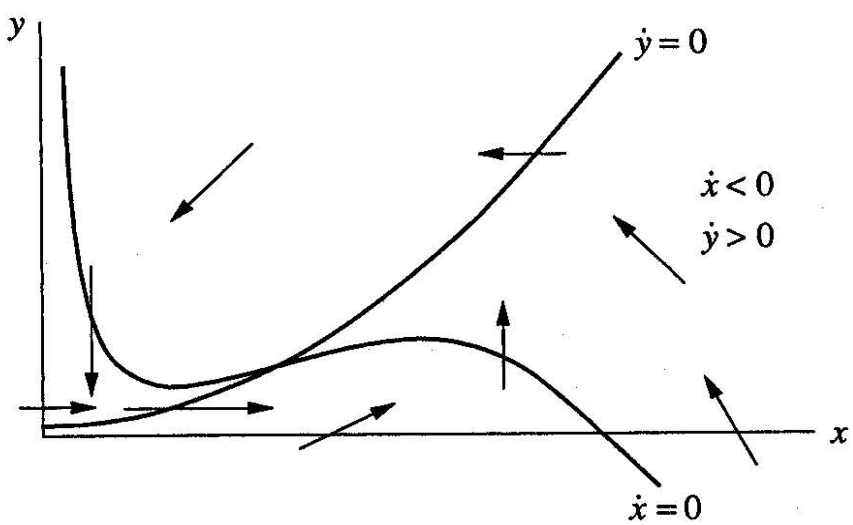


Figure 8.3.1

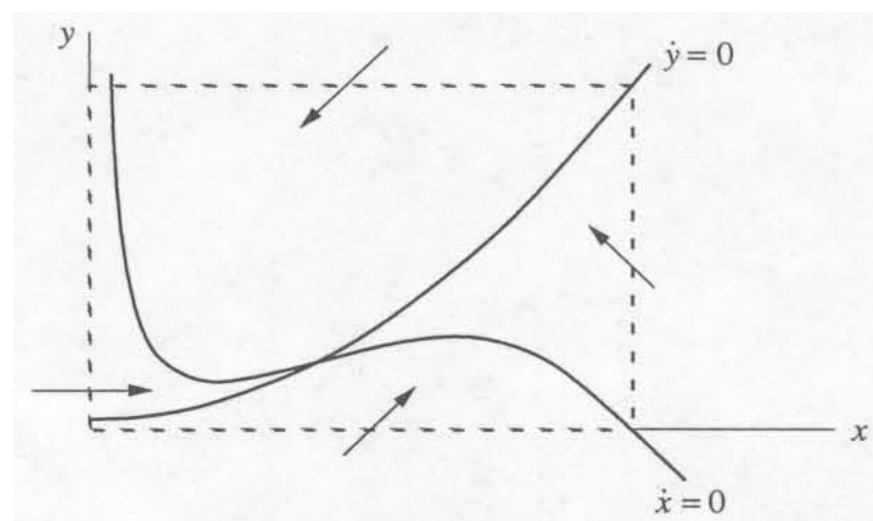


Figure 8.3.2

Oscillating Chemical Reaction

- But there exists fixed point inside box:

$$x^* = a/5 \qquad y^* = 1 + (x^*)^2 = 1 + (a/5)^2.$$

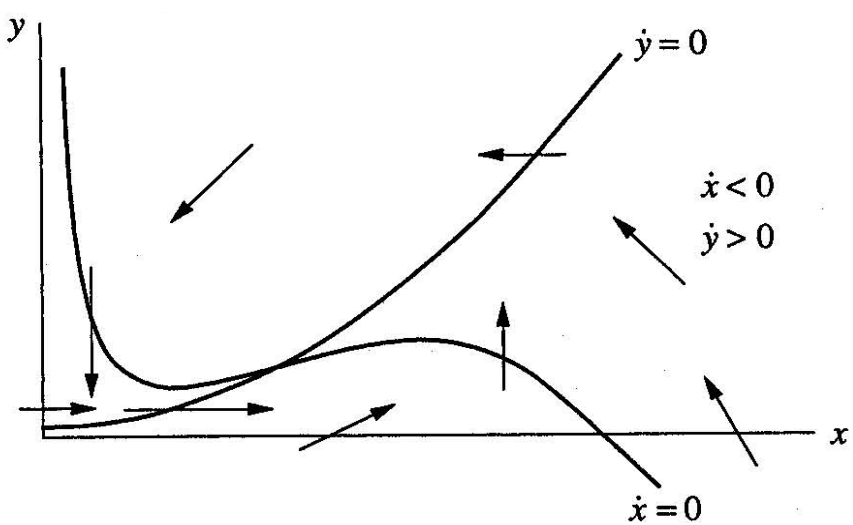
- PB Theorem applies if fixed point is repeller. Since

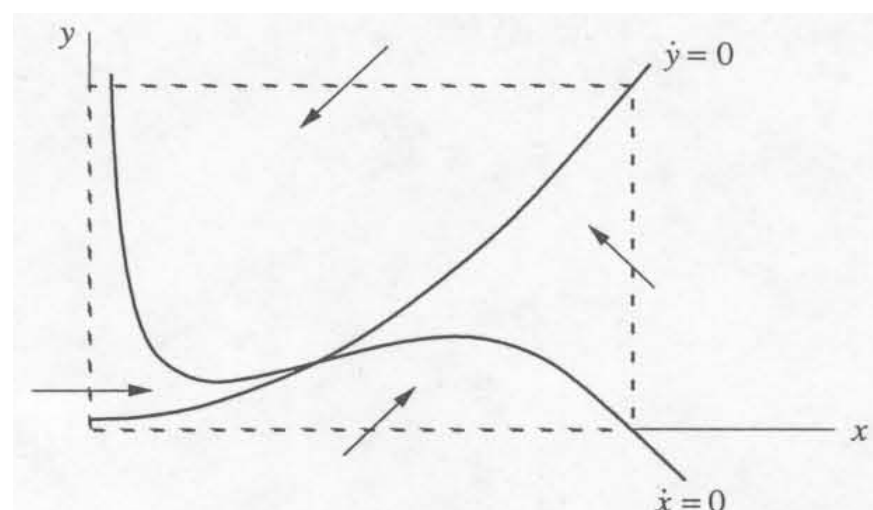
$$\triangle = \frac{5bx^*}{1 + (x^*)^2} > 0, \qquad \tau = \frac{3(x^*)^2 - 5 - bx^*}{1 + (x^*)^2},$$

(x^*, y^*) is repeller if $\tau > 0$, i.e., if

$$b < b_c = 3a/5 - 25/a.$$

- PB Theorem implies existence of a closed orbit in punctured box.





Oscillating Chemical Reaction (Supercritical Hopf)

- As $b \downarrow b_c$, fixed point changes from stable spiral to unstable spiral: Hopf bifurcation.
- Choose $a = 10$, then $b_c = 3.5$.
  - $b > b_c$, all trajectories spiral into stable fixed point.
  - $b < b_c$, they are attracted to stable limit cycle.
- Supercritical: after fixed point loses stability, it is surrounded by stable limit cycle.
- By plotting phase portraits as $b \uparrow b_c$, limit cycle shrinks to a point.

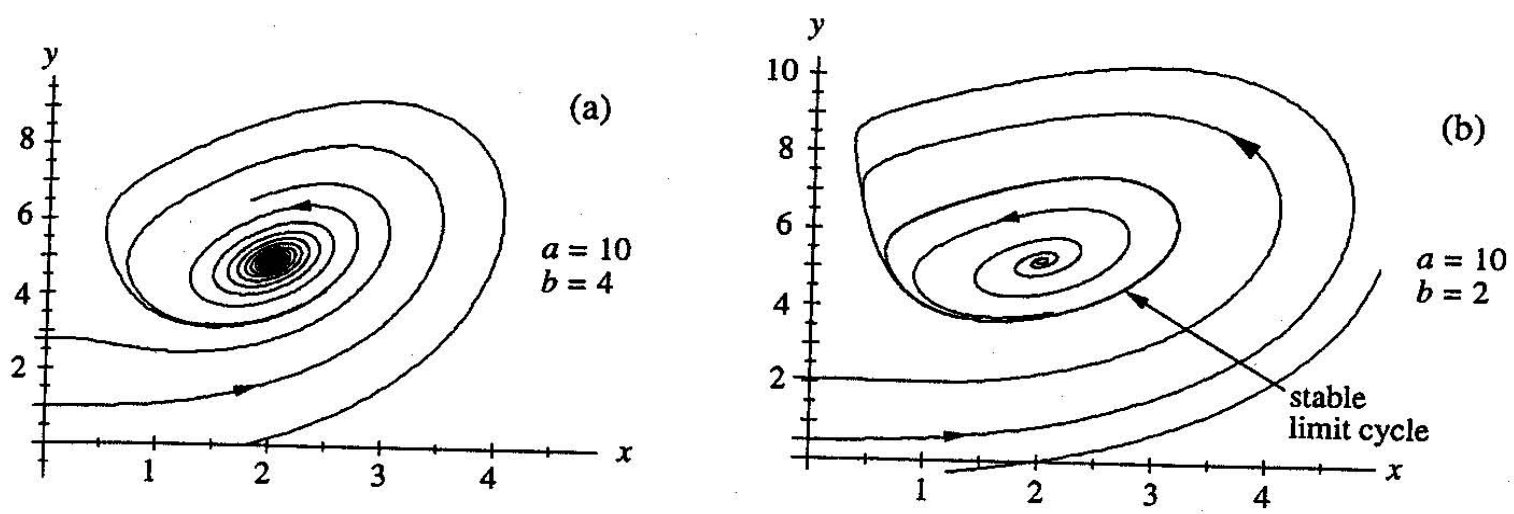


Figure 8.3.3

Approximate Period of Limit Cycle

- Recall: frequency ω is approximated by Im λ at bifurcation.
- Since at b_c,

$$\tau = 0, \qquad \triangle = \frac{5b_c x^*}{1+(x^*)^2} = \frac{15a^2 - 625}{a^2 + 25} > 0,$$

we have (see Figure 8.3.5)

$$\omega = \triangle^{1/2} = [(15a^2 - 625)/(a^2 + 25)], \qquad T = 2\pi/\omega.$$

- Stability diagram (Figure 8.3.4), with Hopf bifurcation locus.

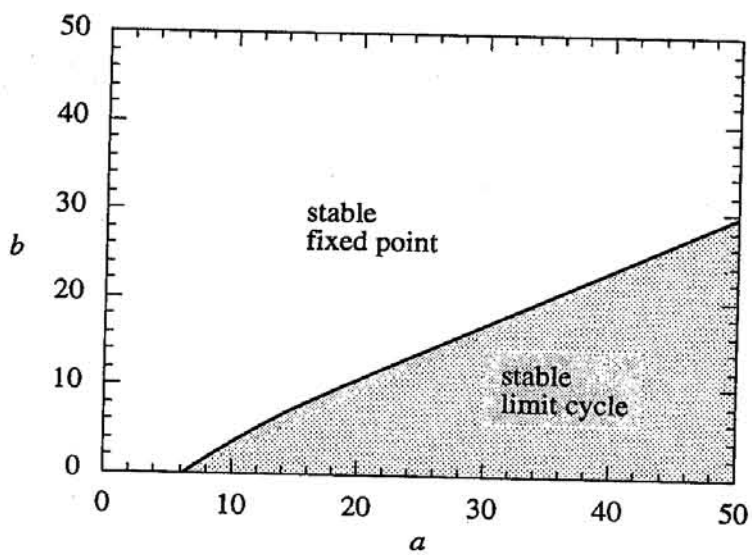


Figure 8.3.4

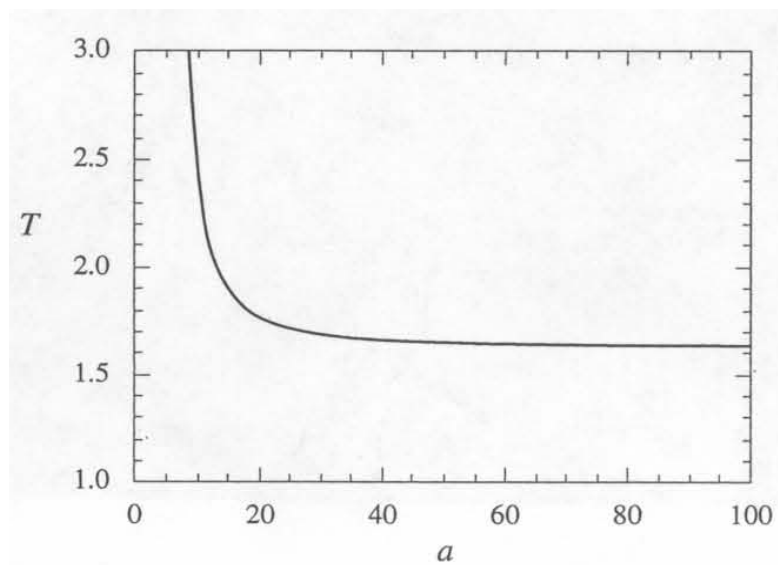


Figure 8.3.5