Oscillating Chemical Reaction

Consider \((x, y)\) are concentrations of \(\Gamma^-, \text{ClO}_2^-\)

\[
\dot{x} = a - x - \frac{4xy}{1 + x^2}, \quad \dot{y} = bx \left(1 - \frac{y}{1 + x^2}\right).
\]

Construct trapping region for Poincaré-Bendixson theorem by sketching nullclines

\[
y = \frac{(a - x)(1 + x^2)}{4x}; \quad x = 0, \quad y = 1 + x^2.
\]

Dashed box is trapping region.
Oscillating Chemical Reaction

But there exists fixed point inside box:

\[ x^* = a/5 \quad \quad y^* = 1 + (x^*)^2 = 1 + (a/5)^2. \]

PB Theorem applies if fixed point is repeller. Since

\[ \Delta = \frac{5bx^*}{1 + (x^*)^2} > 0, \quad \quad \tau = \frac{3(x^*)^2 - 5 - bx^*}{1 + (x^*)^2}, \]

\((x^*, y^*)\) is repeller if \(\tau > 0\), i.e., if

\[ b < b_c = 3a/5 - 25/a. \]

PB Theorem implies existence of a closed orbit in punctured box.
Oscillating Chemical Reaction (Supercritical Hopf)

- As $b \downarrow b_c$, fixed point changes from stable spiral to unstable spiral: Hopf bifurcation.
- Choose $a = 10$, then $b_c = 3.5$.
  - $b > b_c$, all trajectories spiral into stable fixed point.
  - $b < b_c$, they are attracted to stable limit cycle.
- Supercritical: after fixed point loses stability, it is surrounded by stable limit cycle.
- By plotting phase portraits as $b \uparrow b_c$, limit cycle shrinks to a point.
Approximate Period of Limit Cycle

Recall: frequency $\omega$ is approximated by $\text{Im} \lambda$ at bifurcation.

Since at $b_c$,

$$\tau = 0, \quad \Delta = \frac{5b_c x^*}{1 + (x^*)^2} = \frac{15a^2 - 625}{a^2 + 25} > 0,$$

we have (see Figure 8.3.5)

$$\omega = \Delta^{1/2} = \frac{(15a^2 - 625)}{(a^2 + 25)}, \quad T = \frac{2\pi}{\omega}.$$

Stability diagram (Figure 8.3.4), with Hopf bifurcation locus.