**Bifurcations in 2D**

- Bifurcations of fixed point in 1D have analogs in 2D (and higher).
- But action is confined to 1D subspace where bifurcations occur, while flow in other D is attraction or repulsion from 1D subspace.
- **Saddle-node** bifurcation is basic mechanism for creation and destruction of fixed points.
- Prototypical example for **saddle-node** bifurcation in 2D:
  \[
  \dot{x} = \mu - x^2 \quad \dot{y} = -y
  \]
Saddle-Node Bifurcations in 2D

2D system with parameter $\mu$:
\[ \dot{x} = f(x, y, \mu) \quad \dot{y} = g(x, y, \mu). \]

Intersections of nullclines are fixed points.

Saddle-node bifurcations:
- Nullclines pull away as $\mu$ varies, becomes tangent at $\mu = \mu_c$.
- Fixed points approach each other and collide at $\mu = \mu_c$.
- Nullclines pull apart, no intersections. Fixed points disappear.
Example of Saddle-Node Bifurcation in 2D

- Analyse saddle-node bifurcation for genetic control system
  \[ \dot{x} = -ax + y \quad \dot{y} = \left(\frac{x^2}{1 + x^2}\right) - by \]

- To compute \( a_c \), find where fixed points coalesce (\( a_c = 1/2b \)).

- **Unstable manifold** trapped in narrow channel between nullclines.

- **Stable manifold** separate 2 basins of attraction.
  It acts as biochemical switch: gene turns on or off, depending on initial values of \( x, y \).
Transcritical and Pitchfork Bifurcations in 2D

Figure 8.1.5 is similar to 8.1.1. **Saddle node bifurcation** is 1D event. Fixed points slide toward each other.

Prototypical examples of transcritical and Pitchfork bifurcations:
\[
\dot{x} = \mu x - x^2, \quad \dot{y} = -y.
\]
\[
\dot{x} = \mu x - x^3, \quad \dot{y} = -y.
\]

Consider **pitchfork** bifurcation: for \( \mu < 0 \), stable node at origin.

For \( \mu = 0 \), origin is still stable with very slow decay along \( x \)-axis.

For \( \mu > 0 \), origin loses stability, gives birth to 2 new stable fixed points at \((x^*, y^*) = (\pm \sqrt{\mu}, 0)\)
Example of Pitchfork Bifurcation in 2D

- Analyse pitchfork bifurcation for
  \[
  \dot{x} = \mu x + y + \sin x \\
  \dot{y} = x - y.
  \]

- Symmetric w.r.t origin (invariant if \( x \rightarrow -x, y \rightarrow -y \)).

- Origin is fixed point for all \( \mu \). Stable if \( \mu < -2 \), saddle if \( \mu > -2 \) \((\tau = \mu, \Delta = -(\mu + 2))\). Pitchfork may occur at \( \mu_c = -2 \).

- Find 2 new fixed points at \( x^* \approx \sqrt{6(\mu + 2)} \) for \( \mu > -2 \). Supercritical pitchfork occurs at \( \mu_c = -2 \).

- Zero-eigenvalue bifurcations: Saddle-node, transcritical, pitchfork.