Bifurcations in 2D

- Bifurcations of fixed point in 1D have analogs in 2D (and higher).
- But action is confined to 1D subspace where bifurcations occur, while flow in other D is attraction or repulsion from 1D subspace.
- **Saddle-node** bifurcation is basic mechanism for creation and destruction of fixed points.
- Prototypical example for **saddle-node** bifurcation in 2D:

$$\dot{x} = \mu - x^2 \qquad \dot{y} = -y$$

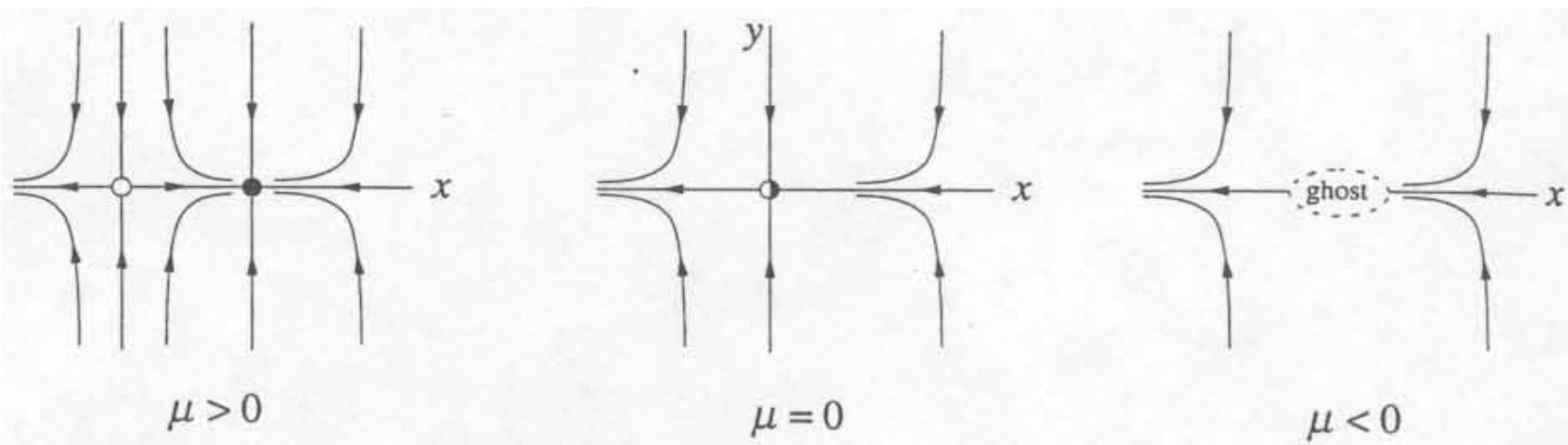


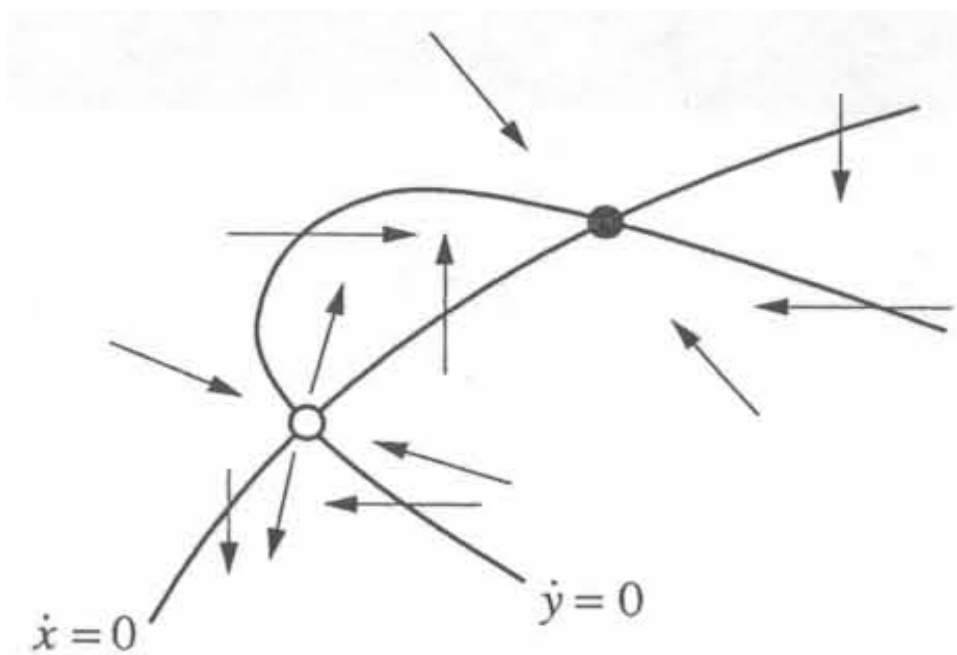
Figure 8.1.1

Saddle-Node Bifurcations in 2D

- 2D system with parameter μ:

$$\dot{x} = f(x, y, \mu) \qquad \dot{y} = g(x, y, \mu).$$

- Intersections of **nullclines** are **fixed points**.
- **Saddle-node** bifurcations:
  - Nullclines pull away as μ varies, becomes tangent at $\mu = \mu_c$.
  - Fixed points approach each other and collide at $\mu = \mu_c$.
  - Nullclines pull apart, no intersections. Fixed points disappear.



Example of Saddle-Node Bifurcation in 2D

- Analyse saddle-node bifurcation for genetic control system

$$\dot{x} = -ax + y \qquad \dot{y} = (x^2/1 + x^2) - by$$

- To compute a_c, find where fixed points coalesce ($a_c = 1/2b$).
- **Unstable manifold** trapped in narrow channel between nullclines.
- **Stable manifold** separate 2 basins of attraction. It acts as **biochemical switch**: gene turns on or off, depending on intial values of x, y.

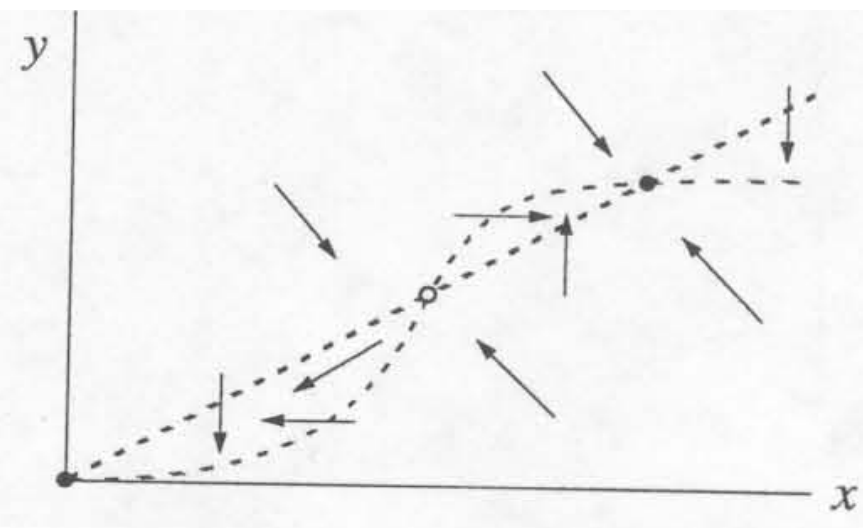


Figure 8.1.4

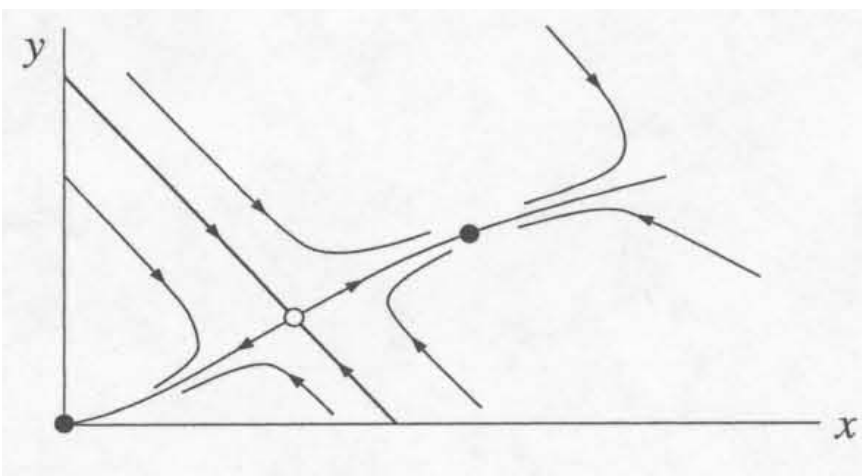


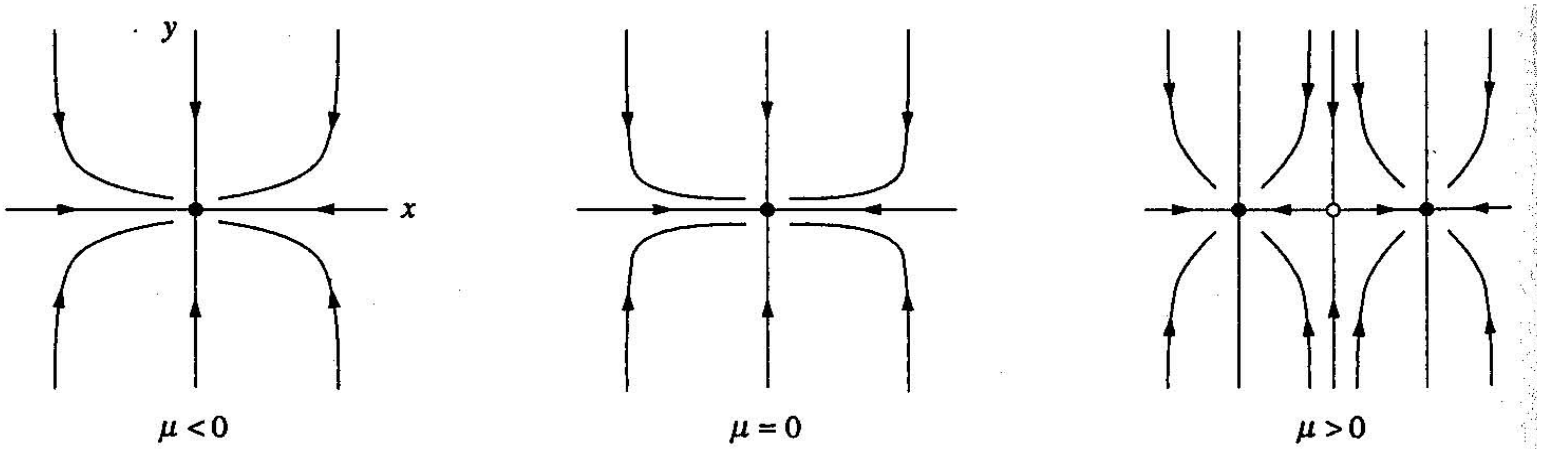
Figure 8.1.5

Transcritical and Pitchfork Bifurcations in 2D

- Figure 8.1.5 is similar to 8.1.1. **Saddle node bifurcation** is 1D event. Fixed points slide toward each other.
- Prototypical examples of transcritical and Pitchfork bifurcations:

$$\dot{x} = \mu x - x^2, \qquad \dot{y} = -y.$$
$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y.$$

- Consider **pitchfork** bifurcation: for $\mu < 0$, stable node at origin.
- For $\mu = 0$, origin is still stable with very slow decay along x-axis.
- For $\mu > 0$, origin loses stability, gives birth to 2 new stable fixed points at $(x^*, y^*) = (\pm\sqrt{\mu}, 0)$



Example of Pitchfork Bifurcation in 2D

- Analyse **pitchfork** bifurcation for

$$\dot{x} = \mu x + y + \sin x \qquad \dot{y} = x - y.$$

- Symmetric w.r.t origin (invariant if $x \to -x, y \to -y$).
- Origin is **fixed point** for all μ. Stable if $\mu < -2$, saddle if $\mu > -2$ ($\tau = \mu, \triangle = -(\mu + 2)$). Pitchfork may occur at $\mu_c = -2$.
- Find 2 new **fixed points** at $x^* \approx \sqrt{6(\mu + 2)}$ for $\mu > -2$. **Supercritical** pitchfork occurs at $\mu_c = -2$
- Zero-eigenvalue bifurcations: Saddle-node, transcritical, pitchfork.