Averaged Equations

- Consider equation $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0.$
  with **2-timing** equations:

$$O(1) \,:\, \partial_{\tau\tau} x_0 + x_0 = 0,$$
$$O(\epsilon) \,:\, \partial_{\tau\tau} x_1 + x_1 = -2\partial_{T\tau} x_0 - h(x_0, \partial_\tau x_0).$$

- Solution for $O(1)$ equation is

$$x_0 = r(T)\cos(\tau + \phi(T)).$$

- Derive eqs for r', ϕ' by insisting RHS has no **resonant terms**.
- Obtain **averaged (slow time) equation** using Fourier analysis

$$r' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta d\theta = \langle h \sin\theta \rangle$$
$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta d\theta = \langle h \cos\theta \rangle$$

where $h = h(r\cos(\tau + \phi), -r\sin(\tau + \phi)) = h(r\cos\theta, -r\sin\theta)$.

Application to Van Der Equation

- Consider van der Pol equation ($\epsilon = 0.1$)

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0,$$

with $x(0) = 1, \dot{x}(0) = 0$.

- Since

$$h = (x^2 - 1)\dot{x} = (r^2 \cos^2\theta - 1)(-r\sin\theta)$$

averaged equations are

$$\begin{aligned} r' &= \langle h \sin\theta \rangle = r\langle \sin^2\theta \rangle - r^3 \langle \cos^2\theta \sin^2\theta \rangle = \tfrac{1}{2}r - \tfrac{1}{8}r^3 \\ r\phi' &= \langle h \cos\theta \rangle = r\langle \sin\theta\cos\theta \rangle - r^3 \langle \sin\theta \cos^3\theta \rangle = 0 \end{aligned}$$

with

$$\begin{aligned} r(0) &= \sqrt{x(0)^2 + \dot{x}(0)^2} = 1, \\ \phi(0) &= \tan^{-1}(\dot{x}(0)/x(0)) - \tau = 0. \end{aligned}$$

Application to Van Der Equation

- **Averaged equations**

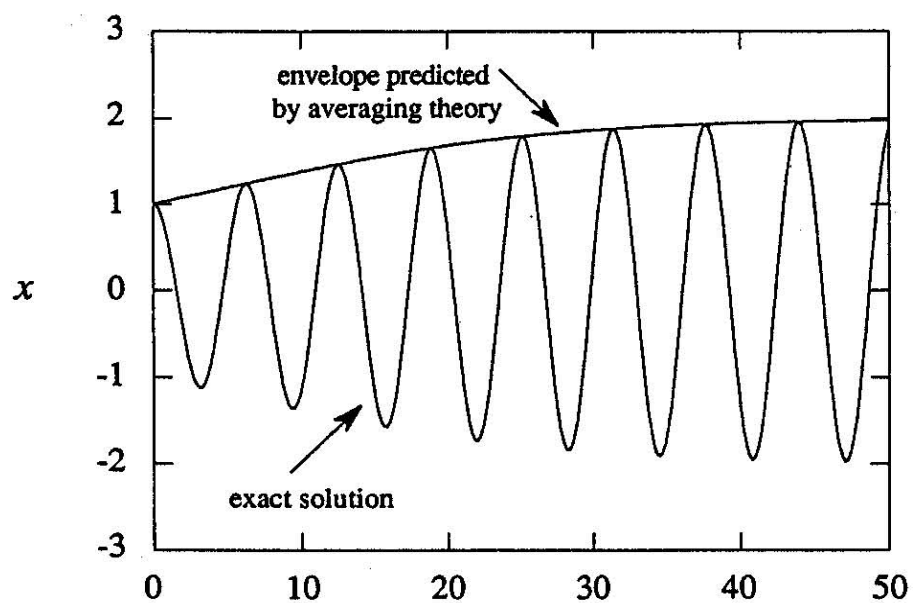
$$r' = \langle h \sin\theta \rangle = \tfrac{1}{2}r - \tfrac{1}{8}r^3, \qquad r(0) = 1,$$
$$r\phi' = \langle h \cos\theta \rangle = 0, \qquad \phi(0) = 0.$$

- Therefore

$$\phi(T) = 0, \qquad r(T) = 2(1 + 3e^{-T})^{-1/2}.$$

and

$$x(t,\epsilon) \approx x_0(\tau, T) + O(\epsilon) = 2(1 + 3e^{-\epsilon t})^{-1/2} \cos t + O(\epsilon).$$



Application to Duffing Equation

- Consider Duffing equation

$$\ddot{x} + x + \epsilon x^3 = 0.$$

- Since

$$h = x^3 = r^3 \cos^3 \theta$$

averaged equations are

$$\begin{aligned} r' &= \langle h \sin\theta \rangle = r^3 \langle \cos^3\theta \sin\theta \rangle = 0 \\ r\phi' &= \langle h \cos\theta \rangle = r^3 \langle \cos^4\theta \rangle = \tfrac{3}{8} r^3 \end{aligned}$$

- Hence, $r(T) = a$ and $\phi' = \frac{3}{8}a^2$.
- **Frequency**

$$\omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT}\frac{dT}{dt} = 1 + \epsilon\phi' = 1 + \tfrac{3}{8}\epsilon a^2 + O(\epsilon^2)$$

- Angular **frequency** depends on **amplitude**.
  Common for nonlinear oscillators, impossible for linear ones.