Consider equation $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$.

with 2-timing equations:

- $O(1): \partial_{TT} x_0 + x_0 = 0$,
- $O(\epsilon): \partial_{TT} x_1 + x_1 = -2\partial_T T x_0 - h(x_0, \partial_T x_0)$.

Solution for $O(1)$ equation is

$$x_0 = r(T) \cos(\tau + \phi(T)).$$

Derive eqs for $r'$, $\phi'$ by insisting RHS has no resonant terms.

Obtain averaged (slow time) equation using Fourier analysis

$$r' = \frac{1}{2\pi} \int_{0}^{2\pi} h(\theta) \sin \theta d\theta = \langle h \sin \theta \rangle,$$

$$r\phi' = \frac{1}{2\pi} \int_{0}^{2\pi} h(\theta) \cos \theta d\theta = \langle h \cos \theta \rangle,$$

where $h = h(r \cos(\tau + \phi), -r \sin(\tau + \phi)) = h(r \cos \theta, -r \sin \theta)$. 


Application to Van Der Equation

Consider van der Pol equation \((\epsilon = 0.1)\)
\[
\ddot{x} + \epsilon (x^2 - 1) \dot{x} + x = 0,
\]
with \(x(0) = 1, \dot{x}(0) = 0\).

Since \(h = (x^2 - 1) \dot{x} = (r^2 \cos^2 \theta - 1)(-r \sin \theta)\)
the averaged equations are
\[
r' = \langle h \sin \theta \rangle = r \langle \sin^2 \theta \rangle - r^3 \langle \cos^2 \theta \sin^2 \theta \rangle = \frac{1}{2}r - \frac{1}{8}r^3
\]
\[
r\phi' = \langle h \cos \theta \rangle = r \langle \sin \theta \cos \theta \rangle - r^3 \langle \sin \theta \cos^3 \theta \rangle = 0
\]
with
\[
r(0) = \sqrt{x(0)^2 + \dot{x}(0)^2} = 1,
\phi(0) = \tan^{-1}(\dot{x}(0)/x(0)) - \tau = 0.
\]
Application to Van Der Equation

Averaged equations

\[
\begin{align*}
    r' &= \langle h \sin \theta \rangle = \frac{1}{2} r - \frac{1}{8} r^3, \quad r(0) = 1, \\
    r\phi' &= \langle h \cos \theta \rangle = 0, \quad \phi(0) = 0.
\end{align*}
\]

Therefore

\[
\phi(T) = 0, \quad r(T) = 2(1 + 3e^{-T})^{-1/2}.
\]

and

\[
x(t, \epsilon) \approx x_0(\tau, T) + O(\epsilon) = 2(1 + 3e^{-\epsilon t})^{-1/2} \cos t + O(\epsilon).
\]
Application to Duffing Equation

Consider Duffing equation

\[ \ddot{x} + x + \epsilon x^3 = 0. \]

Since

\[ h = x^3 = r^3 \cos^3 \theta \]

Averaged equations are

\[ r' = \langle h \sin \theta \rangle = r^3 \langle \cos^3 \theta \sin \theta \rangle = 0 \]
\[ r\phi' = \langle h \cos \theta \rangle = r^3 \langle \cos^4 \theta \rangle = \frac{3}{8} r^3 \]

Hence, \( r(T) = a \) and \( \phi' = \frac{3}{8} a^2 \).

Frequency

\[ \omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT} \frac{dT}{dt} = 1 + \epsilon \phi' = 1 + \frac{3}{8} \epsilon a^2 + O(\epsilon^2) \]

Angular frequency depends on amplitude.
Common for nonlinear oscillators, impossible for linear ones.