Limit Cycle in Liénard System

Consider Liénard equation:
\[ \ddot{x} + f(x)\dot{x} + g(x) = 0. \]

Unit mass subject to damping force and restoring force.

Suppose: \( f(x) \) and \( g(x) \) are \( C^1 \);
1. \( g(-x) = -g(x), \) \( g(x) > 0 \) for \( x > 0 \),
   (restoring force acts like spring)
2. \( f(-x) = f(x) \)
3. \( F(x) = \int_0^x f(u)du \) has 1 positive root at \( x = a; \)
   negative for \( 0 < x < a, \) positive/nondecreasing for \( x > a; \)
   \( F(x) \to \infty \) as \( x \to \infty. \)
   (damping is negative at small \( |x| \), positive at large \( |x| \))

Then the system has a unique, stable limit cycle around \( (0, 0) \).

Apply to van der Pol equation:
\[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \]
Limit Cycle in Van Der Pol Equation

For van der Pol equation

\[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \]

we have

\[ f(x) = \mu(x^2 - 1), \quad g(x) = x. \]

Hence

\[ F(x) = \int_0^x f(u)du = \mu(\frac{1}{3}x^3 - x) = \frac{1}{3}\mu x(x^2 - 3) \]

and \( a = \sqrt{3} \). So, there exists a unique stable limit cycle.
Relaxation Oscillations

- Study **shape** and **period** of limit cycle.
- First consider van der Pol equation with $\mu \gg 1$
  \[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \]

**Relaxation oscillations**: extremely slow buildup followed by sudden discharge (periodic firing of nerve cells).

- After introducing $w = \dot{x} + \mu F(x)$, $F(x) = \frac{1}{3}x^3 - x$.
  
  equation becomes
  \[ \dot{x} = w - \mu F(x), \quad \dot{w} = -x. \]

- Let $y = w/\mu$, then
  \[ \dot{x} = \mu(y - F(x)), \quad \dot{y} = -\frac{1}{\mu}x. \]
Relaxation Oscillations

Consider

\[ \dot{x} = \mu(y - F(x)), \quad \dot{y} = -\frac{1}{\mu}x. \]

Trajectory zaps horizontally onto cubic nullcline, crawls down nullcline to knee, zaps over to other branch, and so on.

If \( y - F(x) \approx O(1) \), \(|\dot{x}| \approx O(\mu) >> 1 \) and \(|\dot{y}| \approx O(\mu^{-1})\).

If \( y - F(x) \approx O(\mu^{-2}) \), \(|\dot{x}| \approx O(\mu^{-1}) \) and \(|\dot{y}| \approx O(\mu^{-1})\).

Jumps require \( \Delta t \approx O(\mu^{-1}) \). Crawls require \( \Delta t \approx O(\mu) \).
**Relaxation Oscillations**

- **Period** $T$ approximated by travel time along 2 slow branches.

- Since
  \[
  \frac{dy}{dt} = F'(x) \frac{dx}{dt} = (x^2 - 1) \frac{dx}{dt},
  \]
  we have
  \[
  dt = -\frac{\mu(x^2 - 1)}{x} dx.
  \]
  Therefore,
  \[
  T = 2 \int_2^1 \frac{-\mu}{x}(x^2 - 1) dx = 2\mu \left[ \frac{x^2}{2} - \ln x \right]_1^2 = \mu[3 - \ln 2].
  \]