Limit Cycle in Liénard System

- Consider Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

  Unit mass subject to **damping force** and **restoring force**.

- Suppose: $f(x)$ and $g(x)$ are C^1;

  1. $g(-x) = -g(x)$, $g(x) > 0$ for $x > 0$,
     (**restoring force** acts like spring)
  2. $f(-x) = f(x)$
  3. $F(x) = \int_0^x f(u)du$ has 1 positive root at $x = a$;
     negative for $0 < x < a$, positive/nondecreasing for $x > a$;
     $F(x) \to \infty$ as $x \to \infty$.
     (**damping** is negative at small $|x|$, positive at large $|x|$)

  Then the system has a unique, **stable limit cycle** around $(0, 0)$.

- Apply to van der Pol equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

Limit Cycle in Van Der Pol Equation

- For van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

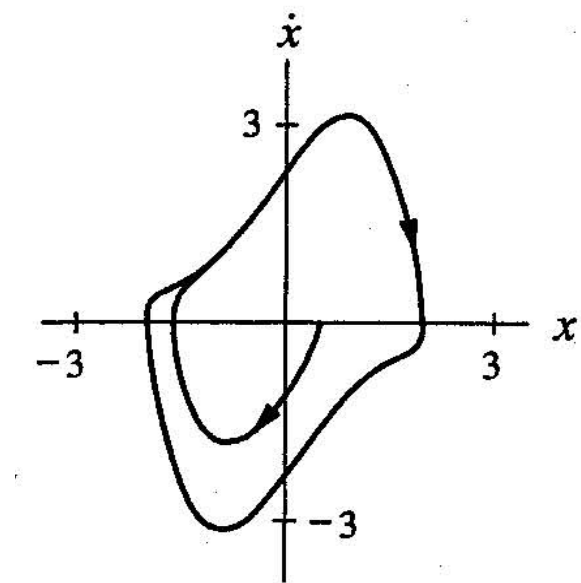
we have

$$f(x) = \mu(x^2 - 1), \qquad g(x) = x.$$

- Hence

$$F(x) = \int_0^x f(u)du = \mu(\tfrac{1}{3}x^3 - x) = \frac{1}{3}\mu x(x^2 - 3)$$

and $a = \sqrt{3}$. So, there exists a unique **stable limit cycle**.



Relaxation Oscillations

- Study **shape** and **period** of limit cycle.
- First consider van der Pol equation with $\mu >> 1$

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

- **Relaxation oscillations**: extremely slow buildup followed by sudden discharge (periodic firing of nerve cells).
- After introducing

$$w = \dot{x} + \mu F(x), \qquad F(x) = \frac{1}{3}x^3 - x.$$

equation becomes

$$\dot{x} = w - \mu F(x), \qquad \dot{w} = -x.$$

- Let $y = w/\mu$, then

$$\dot{x} = \mu(y - F(x)), \qquad \dot{y} = -\frac{1}{\mu}x.$$

Relaxation Oscillations

- Consider
$$\dot{x} = \mu(y - F(x)), \qquad \dot{y} = -\frac{1}{\mu}x.$$
- Trajectory zaps horizontally onto cubic **nullcline**, crawls down **nullcline** to knee, zaps over to other branch, and so on.
- If $y - F(x) \approx O(1)$, $|\dot{x}| \approx O(\mu) >> 1$ and $|\dot{y}| \approx O(\mu^{-1})$.
- If $y - F(x) \approx O(\mu^{-2})$, $|\dot{x}| \approx O(\mu^{-1})$ and $|\dot{y}| \approx O(\mu^{-1})$.
- **Jumps** require $\triangle t \approx O(\mu^{-1})$. **Crawls** require $\triangle t \approx O(\mu)$.

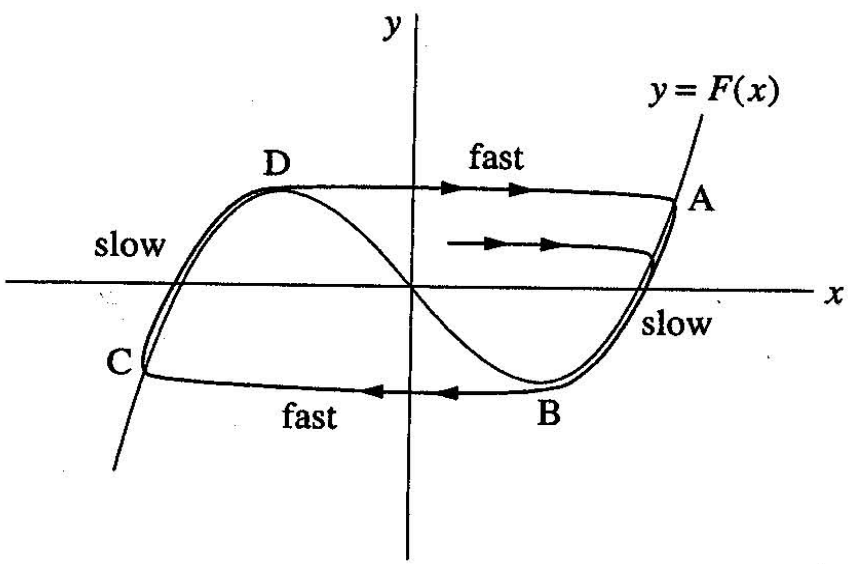


Figure 7.5.1

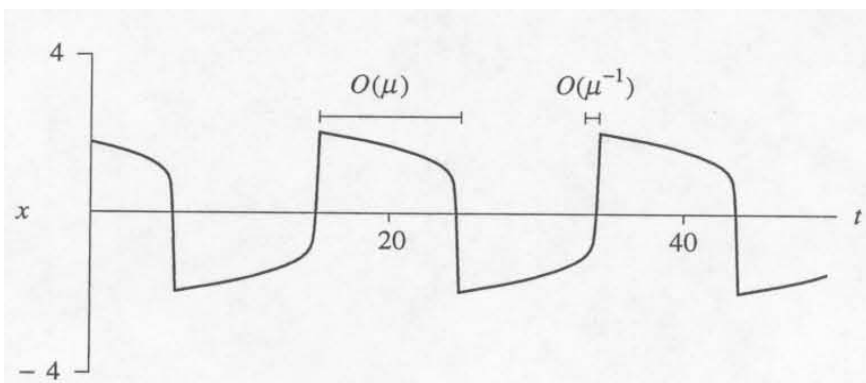


Figure 7.5.2

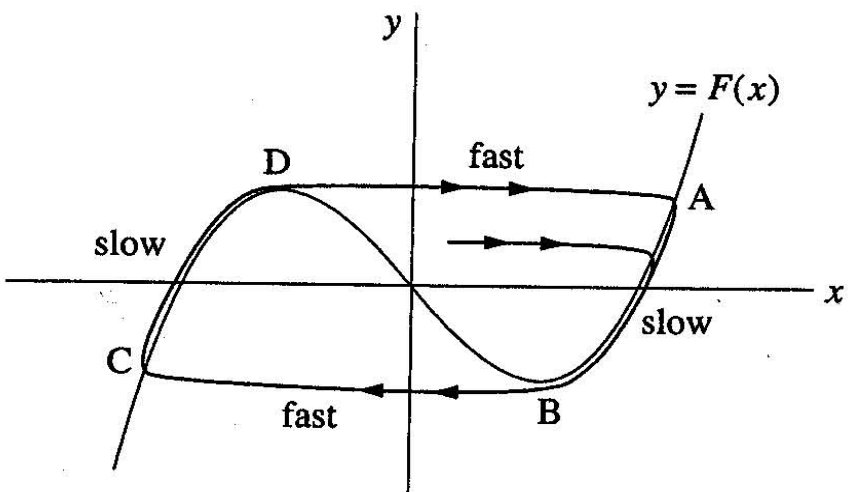
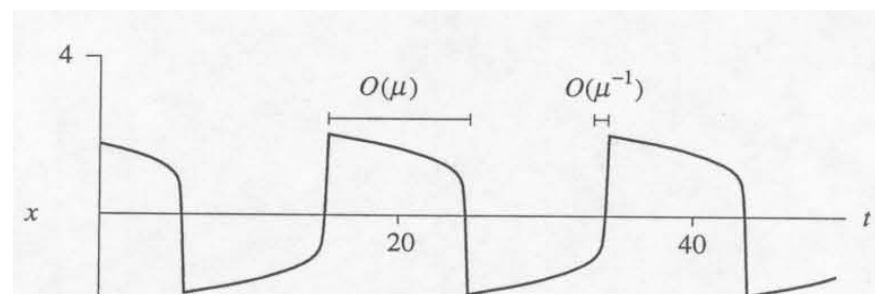
Relaxation Oscillations

- **Period** T approximated by travel time along 2 slow branches.
- Since

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt} = (x^2 - 1)\frac{dx}{dt},$$

we have

$$dt = -\frac{\mu(x^2 - 1)}{x}dx.$$

Therefore,

$$T = 2\int_2^1 \frac{-\mu}{x}(x^2 - 1)dx = 2\mu\left[\frac{x^2}{2} - lnx\right]_1^2 = \mu[3 - ln2].$$