**Index Theory**

- Provide **global information** about phase portrait.
- Let $\dot{x} = f(x)$ be **vector field**, $C$ **simple closed curve** that doesn’t pass through any **fixed point**, then **index** of $C$ w.r.t. $f$:
  \[
  I_C = \frac{1}{2\pi} [\phi]_C
  \]
  where $[\phi]_C$ is **net change** in $\phi$ over one circuit.

- Example (figure 6.8.3): $I_C = +1$.  

![Figure 6.8.1](image1) ![Figure 6.8.3](image2)
**Index Theory (Examples)**

- **Index** of $C$ w.r.t. $f$:
  \[ I_C = \frac{1}{2\pi} [\phi]_C \]
  where $[\phi]_C$ is **net change** in $\phi$ over one circuit.

- **Example** (figure 6.8.4): $I_C = -1$.

- Consider vector field $\dot{x} = x^2y, \dot{y} = x^2 - y^2, C : x^2 + y^2 = 1$: $I_C = 0$. 

![Figure 6.8.4](image1.png)

![Figure 6.8.5](image2.png)
Properties of the Index

- Suppose $C$ continuously **deformed** into $C'$ without passing through fixed point, then $I_C = I_{C'}$.
- If $C$ doesn’t enclose any **fixed points**, then $I_C = 0$.
- If reverse all arrows in vector field ($t \rightarrow -t$), index unchanged.
- Suppose $C$ is a **trajectory**, then $I_C = +1$.

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**Figure 6.8.6**

**Figure 6.8.7**
Index of a Point

- Suppose \( x^* \) is isolated fixed point, index \( I \) of \( x^* \) is \( I_C \) where \( C \) is any closed curve that encloses \( x^* \) and no other fixed points.
- Stable and unstable node (\( I = +1 \)). Saddle point (\( I = -1 \)).
- **Theorem**: If \( C \) surrounds \( n \) isolated fixed points, then

\[
I_C = I_1 + \ldots + I_n.
\]
- **Theorem**: Any closed orbit in phase plane must enclose fixed points whose *indices* sum to +1.

![Diagram](image-url)
Consequences of Theorem 6.8.2

- Theorem 6.8.2: Any closed orbit in phase plane must enclose fixed points whose *indices* sum to +1.
- At least one *fixed point* inside any closed orbit in phase plane.
- If only one fixed point inside, it cannot be *saddle point*.
- Rule out *closed trajectories* (for Rabbit vs Sheep).

![Figure 6.8.9]
Pendulum Equation

If no damping and external driving, motion of pendulum

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0. \]

With \( \tau = \omega t \) (dimensionless time) where \( \omega = \sqrt{g/L} \) (frequency),

\[ \dot{\theta} = \nu, \quad \dot{\nu} = -\sin \theta. \]

where \( \nu \) is (dimensionless) angular velocity.

\[ \frac{d^2 \theta}{dt^2} \]

\[ g \]

\[ L \]

\[ \theta \]

\[ m \]

\[ \text{Figure 6.7.1} \]
Nonlinear Centers and Saddle Points

- Fixed points: \((k\pi, 0)\).
- **Nonlinear center** at \((0, 0)\). **Reversible** \((\tau \rightarrow -\tau, \nu \rightarrow -\nu)\).
- **Nonlinear center** at \((0, 0)\). **Conservative** \((E = \frac{1}{2}\nu^2 - \cos \theta)\).
- **Saddle** at \((\pi, 0)\). \(\lambda = \pm 1\) with \(v = (1, \pm 1)\).

![Figure 6.7.3](image-url)
Adding Linear Damping

Equation becomes

\[ \ddot{\theta} + b\dot{\theta} + \sin \theta = 0. \]

Centers become stable spirals. Saddles remain saddles.

Change in energy along a trajectory

\[ \frac{dE}{d\tau} = \dot{\theta}(\ddot{\theta} + \sin \theta) = -b\dot{\theta}^2 \leq 0. \]

Whirl clockwise. Settle into small oscillation. Come to rest.