Fixed Points and Linearization

Suppose \((x^*, y^*)\) is fixed point, the linearized system is

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

where \(u = x - x^*, v = y - y^*\). Jacobian matrix.

If fixed point for linearized system is not one of borderline cases, linearized system give a qualitatively correct picture near \((x^*, y^*)\).

Borderline cases can be altered by small nonlinear terms.
Example 6.3.1

Consider

\[
\begin{align*}
\dot{x} &= -x + x^3, \\
\dot{y} &= -2y.
\end{align*}
\]

3 fixed points: (0, 0) (stable node), (1, 0), (−1, 0) (saddles) by analyzing linearized system.

Not borderline cases. Fixed points for nonlinear system is similar to linearized system.

![Diagram of phase portrait showing fixed points and trajectories](image-url)
Example 6.3.2

Consider

\[
\begin{align*}
\dot{x} &= -y + ax(x^2 + y^2), \\
\dot{y} &= x + ay(x^2 + y^2).
\end{align*}
\]

Linearized system predicts: \((0, 0)\) is a center for all \(a\).

In polar coordinates

\[
\begin{align*}
\dot{r} &= ar^3, \\
\dot{\theta} &= 1.
\end{align*}
\]

In fact, \((0, 0)\) is a spiral (stable if \(a < 0\), unstable if \(a > 0\)).

Stars and degenerate nodes can be altered by small nonlinearities, but their stability doesn’t change.
If only interested in stability

- **Robust Cases:**
  - Repellers (sources): Re(\(\lambda_1\)), Re(\(\lambda_2\)) > 0.
  - Attractors (sinks): Re(\(\lambda_1\)), Re(\(\lambda_2\)) < 0.
  - Saddles: \(\lambda_1 > 0, \lambda_2 < 0\).

- **Marginal Cases:**
  - Centers: both eigenvalues are pure imaginary.
  - Higher-order and Non-isolated fixed points: at least one eigenvalues is zero.

If Re(\(\lambda\)) \(\neq 0\) for both eigenvalues, fixed point is **hyperbolic**.

**Hartman-Grobman Theorem:** Local phase portrait near a hyperbolic fixed point is topologically equivalent to phase portrait of its linearized system.

A phase portrait is **structurally stable** if its topology cannot be changed by an arbitrarily small perturbation to the vector field.
Lotka-Volterra Model of Competition

- Consider Rabbit ($x$) vs Sheep ($y$)
  \[
  \dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - y - x).
  \]

- Find fixed points: $(0, 0), (0, 2), (3, 0), (1, 1)$.

- Compute Jacobian matrix and classify fixed points: $(0, 0)$ (unstable node), $(0, 2), (3, 0)$ (stable nodes), $(1, 1)$ (saddle).

- Draw phase portrait. Basic of attraction for fixed points.

- Principle of competitive exclusion.