Dirac Structures and the Legendre Transformation for Implicit Lagrangian and Hamiltonian Systems

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Background: Network Modeling

- In conjunction with *network modeling of complex physical systems*, the idea of *interconnections*, first proposed by *G. Kron (1939)*, is a very useful tool that enables us to treat an original system as a *network* of an aggregation of torn apart subsystems or elements.

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- In conjunction with *network modeling of complex physical systems*, the idea of *interconnections*, first proposed by *G. Kron (1939)*, is a very useful tool that enables us to treat an original system as a *network* of an aggregation of torn apart subsystems or elements.

- Especially, the interconnections play an essential role in modeling physical systems interacting with various energy fields such as electro-mechanical systems *(Kron,1963)*, bio-chemical reaction systems *(Kachalsky, Oster and Perelson,1970)*, etc.

What is Interconnection ?

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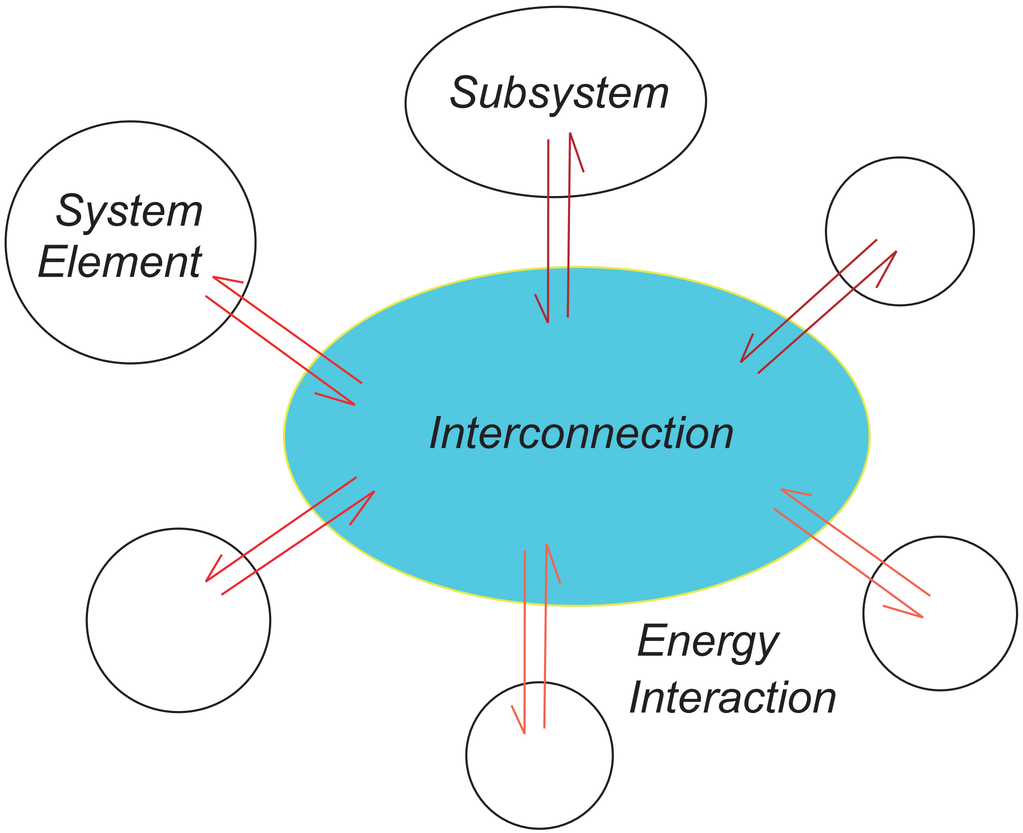
- The interconnection represents how subsystems or elements are energetically interacted with each other; in other words, it plays a role in regulating energy flow between subsystems and elements.



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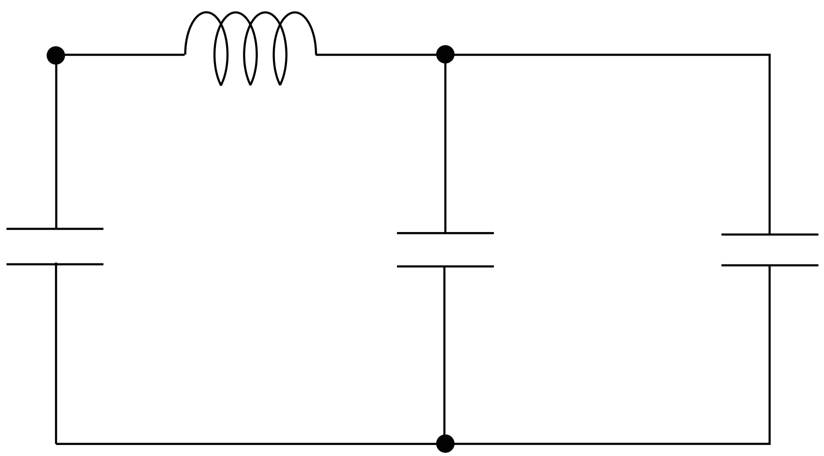
- The interconnection in electric circuits is a typical example, in which we can *literally* see *how system components are interconnected*.



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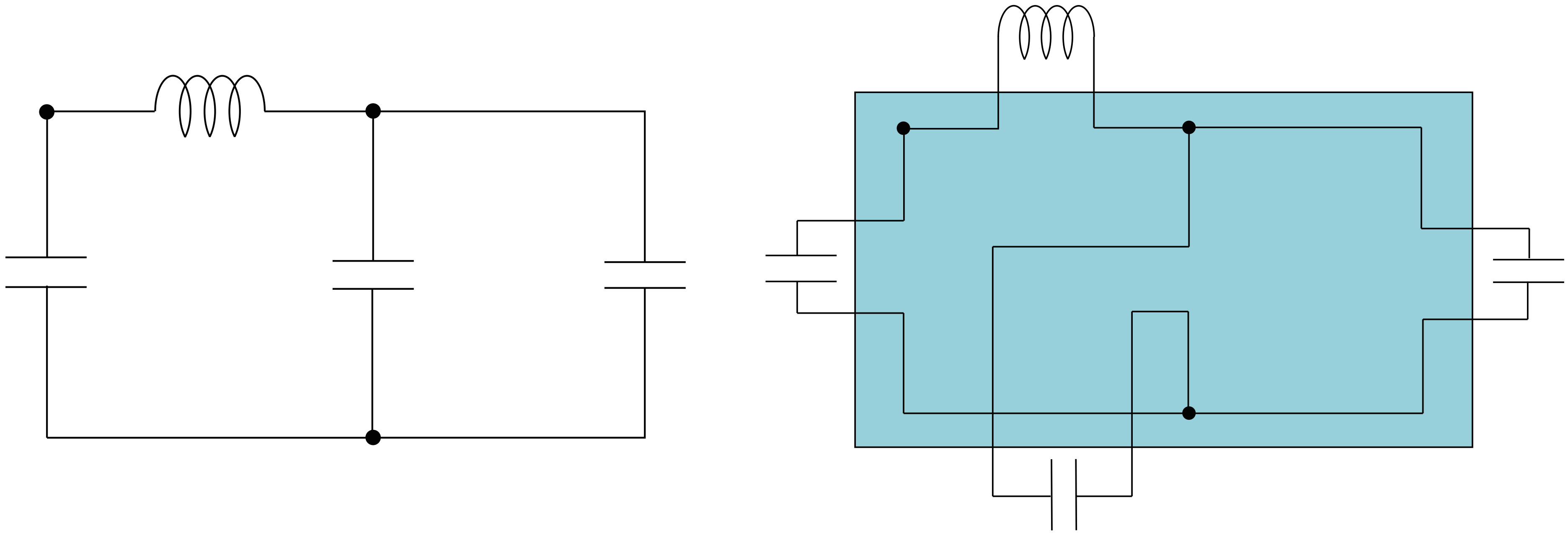

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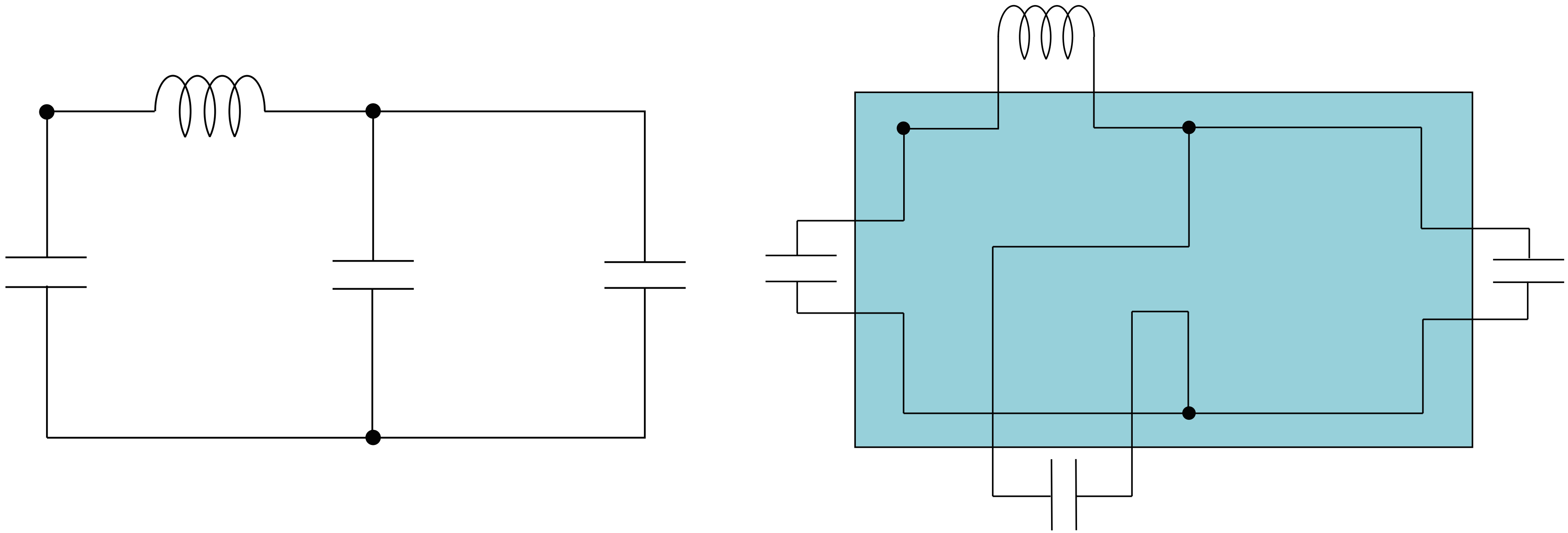

- The interconnection of L-C circuits was shown to be represented by *Dirac structures* by *van der Schaft and Maschke (1995)* and *Bloch and Crouch (1997)*.

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- *Courant and Weinstein (1989, 1991)* developed a notion of Dirac structures that include "symplectic and Poisson structures", inspiring from *Dirac's theory of constraints*.
- An *almost Dirac structure on a manifold M* is defined by, for each $x \in M$,

$$D(x) \subset T_xM \times T_x^*M \text{ such that } D(x) = D^{\perp}(x),$$

where

$$D^{\perp}(x) = \{(\bar{v}_x, \bar{\alpha}_x) \in T_xM \times T_x^*M \mid \langle \alpha_x, \bar{v}_x \rangle + \langle \bar{\alpha}_x, v_x \rangle = 0, \forall (v_x, \alpha_x) \in D(x)\}.$$

- We call D a ***Dirac structure*** on M if

$$\langle \pounds_{X_1}\alpha_2, X_3\rangle + \langle \pounds_{X_2}\alpha_3, X_1\rangle + \langle \pounds_{X_3}\alpha_1, X_2\rangle = 0$$

for all $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in D$.

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- The bundle map $\Omega^\flat : TP \longrightarrow T^*P$ associated to a *two-form Ω on P* defines a Dirac structure on P as

$$D_P = \mathrm{graph}\, \Omega^\flat \subset TP \oplus T^*P.$$

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$$D_P = \mathrm{graph}\,\Omega^\flat \subset TP \oplus T^*P.$$

- The bundle map $B^\sharp : T^*P \to TP$ associated to a *Poisson structure B on P* defines a Dirac structure on P as

$$D_P = \mathrm{graph}\,B^\sharp \subset TP \oplus T^*P.$$

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$$(X, \mathbf{d}H) \in D_P.$$

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$$(X, \mathbf{d}H) \in D_P.$$

In the case that $P = T^*Q$, the coordinate expression of the implicit Hamiltonian system is given by

$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p^i} \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a(q) \end{pmatrix},$$

$$0 = \omega_i^a(q) \, \frac{\partial H}{\partial p^i}.$$

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- Dirac structures have not been enough investigated from the Lagrangian side, although Dirac's theory of constraints started from a *degenerate Lagrangian*. Recently, a notion of *implicit Lagrangian systems*, has been developed by *Yoshimura and Marsden (2003)*.
- For *degenerate cases*, we need to *do* ***"slowly and carefully"*** *the Legendre transform*. A *generalized Legendre transformation* was developed by *Tulczyjew (1974)* and *Maxwell-Vlasov equations* were investigated by *Euler-Poincaré equations* in the context of the generalized Legendre transform with symmetry by *Cendra, Holm, Hoyle and Marsden (1998)*.

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- Both implicit Lagrangian and Hamiltonian systems are equivalent even in degenerate cases?

Our Goals are to Answer these Questions!

Induced Dirac Structures

- Consider nonholonomic constraints which are given by a regular distribution

$$\Delta_Q \subset TQ.$$

Let $\pi_Q : T^*Q \to Q$ be the canonical projection and its tangent map is given by

$$T\pi_Q : TT^*Q \to TQ;$$
$$(q, p, \delta q, \delta p) \mapsto (q, \delta q).$$

Lift up the distribution Δ_Q on Q to T^*Q such that

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q.$$

Induced Dirac Structures

- Define a skew-symmetric bilinear form Ω_{Δ_Q} by

$$\Omega_{\Delta_Q} = \Omega \mid_{\Delta_{T^*Q} \times \Delta_{T^*Q}} .$$

An ***induced Dirac structure*** D_{Δ_Q} ***on*** T^*Q is defined by, for each $(q, p) \in T^*Q$,

$$D_{\Delta_Q}(q,p) = \{(v, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid$$

$$v \in \Delta_{T^*Q}(q,p), \ \text{ and } \alpha(w) = \Omega_{\Delta_Q}(v, w)$$

$$\text{for all } \quad w \in \Delta_{T^*Q}(q,p)\}.$$

Symplectomorphisms

- There are natural diffeomorphisms as

$$(1)\quad \kappa_Q : TT^*Q \to T^*TQ;\ (q,p,\delta q,\delta p) \mapsto (q,\delta q,\delta p,p)$$

$$(2)\quad \Omega^\flat : TT^*Q \to T^*T^*Q;\ (q,p,\delta q,\delta p) \mapsto (q,p,-\delta p,\delta q)$$

Then, define the diffeomorphism by

$$\gamma_Q = \Omega^\flat \circ (\kappa_Q)^{-1} : T^*TQ \to T^*T^*Q,$$

which is given in coordinates by

$$(q,\delta q,\delta p,p) \mapsto (q,p,-\delta p,\delta q),$$

which preserves the symplectic form Ω_{TT^*Q} on TT^*Q:

$$\Omega_{TT^*Q} = dq \wedge d\delta p + d\delta q \wedge dp.$$

Dirac Differential

- Let $L : TQ \to \mathbb{R}$ be a Lagrangian (possibly degenerate) and $\mathbf{d}L : TQ \to T^*TQ$ is given by

$$\mathbf{d}L = \left(q, v, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial v} \right).$$

Define the ***Dirac differential*** of L by

$$\mathfrak{D}L = \gamma_Q \circ \mathbf{d}L : TQ \to T^*T^*Q.$$

In coordinates,

$$\mathfrak{D}L = \left(q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v \right),$$

where we have the Legendre transform $p = \partial L / \partial v$.

Implicit Lagrangian Systems

- An ***implicit Lagrangian system*** is a triple (L, Δ_Q, X) which satisfies, for each $(q,v) \in \Delta_Q$,

$$(X(q,p), \mathfrak{D}L(q,v)) \in D_{\Delta_Q}(q,p),$$

where $(q,p) = \mathbb{F}L(q,v)$.

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- Since the canonical two-form Ω is locally given by

$$\Omega\left((q,p,u_1,\alpha_1),(q,p,u_2,\alpha_2)\right) = \langle \alpha_2, u_1 \rangle - \langle \alpha_1, u_2 \rangle\,,$$

the Dirac structure is locally expressed by

$$\begin{aligned} D_{\Delta_Q}(q,p) = \{((q,p,\dot{q},\dot{p}),(q,p,\alpha,w)) \mid \dot{q} \in \Delta(q), \\ w = \dot{q}, \quad \text{and} \quad \alpha + \dot{p} \in \Delta^{\circ}(q)\}\,. \end{aligned}$$

Implicit Lagrangian Systems

- Since $X(q,p) = (q, p, \dot{q}, \dot{p})$ and $\mathfrak{D}L = \left(q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v\right)$, it reads from $(X, \mathfrak{D}L) \in D_{\Delta_Q}$ that, for each $v \in \Delta(q)$,

$$\left\langle -\frac{\partial L}{\partial q}, u \right\rangle + \langle v, \alpha \rangle = \langle \alpha, \dot{q} \rangle - \langle \dot{p}, u \rangle \,,$$

for all $u \in \Delta(q)$, all α and with $p = \partial L / v$.

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Thus, one can obtain the coordinate expression of ***implicit Lagrangian systems:***

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^{\circ}(q), \quad \dot{q} = v, \quad p = \frac{\partial L}{\partial v}, \quad \dot{q} \in \Delta(q).$$

Hamilton-Pontryagin Principle

- Given a Lagrangian $L : TQ \to \mathbb{R}$ (possibly degenerate). By regarding the second-order condition

$$\dot{q} = v$$

as a constraint, we define the action integral by

$$\begin{aligned}\mathfrak{S}(q, v, p) &= \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\}\, dt \\ &= \int_{t_1}^{t_2} \{p(t) \cdot \dot{q}(t) - E(q(t), v(t), p(t))\}\, dt,\end{aligned}$$

where $E(q, v, p) = p \cdot v - L(q, v)$ is the generalized energy on $TQ \oplus T^*Q$.

□ Keeping the endpoints of $q(t)$ fixed, the stationary condition for the action functional is

$$\delta \int_{t_1}^{t_2} \{L(q,v) + p\,(\dot{q} - v)\}\, dt$$

$$= \int_{t_1}^{t_2} \left\{ \left(-\dot{p} + \frac{\partial L}{\partial q} \right) \delta q + \left(-p + \frac{\partial L}{\partial v} \right) \delta v + (\dot{q} - v)\delta p \right\} dt$$

$$= 0,$$

which is satisfied for all δq, δv and δp.

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We obtain ***implicit Euler-Lagrange equations:***

$$\dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}, \quad \dot{q} = v.$$

Lagrange-d'Alembert-Pontryagin Principle

- Let $\Delta_Q \subset TQ$ be a distribution. The ***Lagrange-d'Alembert-Pontryagin Principle*** is given by

$$\int_{t_1}^{t_2} \left\{ \left(-\dot{p} + \frac{\partial L}{\partial q} \right) \delta q + \left(-p + \frac{\partial L}{\partial v} \right) \delta v + \left(\dot{q} - v \right) \delta p \right\} dt = 0$$

for all chosen $\delta q \in \Delta_Q(q)$, δv, δp, and with $v \in \Delta_Q(q)$.

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Then, we obtain an implicit Lagrangian system as

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^{\circ}(q), \quad \dot{q} = v, \quad p = \frac{\partial L}{\partial v}, \quad \text{and} \quad \dot{q} \in \Delta(q).$$

Example: Point Vortices

- Consider a system with a degenerate Lagrangian:

$$L(q, v) = \langle \alpha_i(q), v^i \rangle - h(q),$$

which arises in ***point vortices and KdV equations*** (Marsden and Ratiu (1999)).

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In the context of implicit Lagrangian systems, we have

$$\dot{q}^i = v^i,$$
$$\dot{p}_i = \frac{\partial L}{\partial q^i} = \frac{\partial \alpha_j(q)}{\partial q^i} v^j - \frac{\partial h(q)}{\partial q^i},$$
$$p_i = \frac{\partial L}{\partial v^i} = \alpha_i(q).$$

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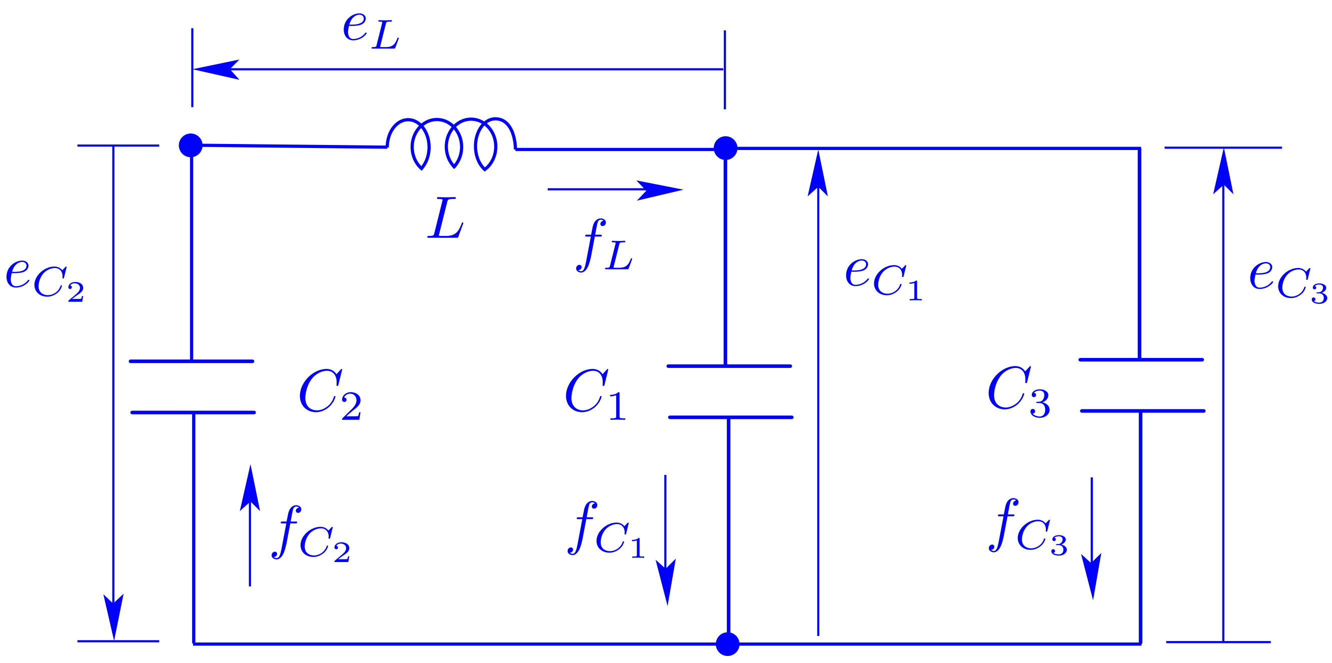

$$\text{charges: } q = (q_L, q_{C_1}, q_{C_2}, q_{C_3}) \in W,$$

$$\text{currents: } f = (f_L, f_{C_1}, f_{C_2}, f_{C_3}) \in T_qW,$$

$$\text{voltages: } e = (e_L, e_{C_1}, e_{C_2}, e_{C_3}) \in T_q^*W.$$

□ The KCL constraint for currents is given by

$$\Delta_q = \{ f \in T_q W \mid \langle \omega^a, f \rangle = 0, \quad a = 1, 2 \},$$

where

$$\omega^1 = -dq_L + dq_{C_2} \text{ and } \omega^2 = -dq_{C_1} + dq_{C_2} - dq_{C_3}.$$

The lifted distribution on T^*W is given by

$$\Delta_{T^*W} = \left\{ X_{(q,p)} = (q, p, \dot{q}, \dot{p}) \mid q \in U, \; \dot{q} \in \Delta_q \right\}$$

and an induced Dirac structure on T^*W is defined as

$$D_\Delta(q,p) = \{ ((q,p,\dot{q},\dot{p}), (q,p,\alpha,w)) \mid \dot{q} \in \Delta_q,$$
$$w = \dot{q}, \text{ and } \alpha + \dot{p} \in \Delta_q^\circ \}.$$

- The Lagrangian of the L-C circuit is given by

$$\mathcal{L}(q,f) = T_q(f) - V(q)$$
$$= \frac{1}{2}L\left(f_L\right)^2 - \frac{1}{2}\frac{(q_{C_1})^2}{C_1} - \frac{1}{2}\frac{(q_{C_2})^2}{C_2} - \frac{1}{2}\frac{(q_{C_3})^2}{C_3}$$

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The image of Δ, namely, $P = \mathbb{F}L(\Delta) \subset T^*W$ indicates the primary constraint set as

$$p_L = L\,f_L, \quad p_{C_1} = p_{C_2} = p_{C_3} = 0.$$

The Dirac differential of $\mathcal{L}$ is denoted by

$$\mathfrak{D}\mathcal{L}(q,f) = \left(0, \frac{q_{C_1}}{C_1}, \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, f_L, f_{C_1}, f_{C_2}, f_{C_3}\right).$$

□ The L-C circuit satisfies the condition

$$(X, \mathfrak{D}\mathcal{L}) \in D_\Delta.$$

Thus, the L-C circuit can be represented by

$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial \mathcal{L}}{\partial q^i} \\ v^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a(q) \end{pmatrix},$$

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- *Q: How can we go to the Hamiltonian side in degenerate cases ?*

  A: We can go to the Hamitlonian side by incorporating primary constraints.

Generalized Legendre Transform

- The ***constraint momentum space*** is defined by

$$P = \mathbb{F}L(\Delta_Q) \subset T^*Q,$$

where we suppose that $\dim P_q = k \leq n$ at each $q \in Q$ and P_q is given by the ***primary constraints*** as

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$$P_q = \left\{ p \in T_q^* Q \mid \phi_A(q, p) = 0,\ A = k+1, ..., n \right\},$$

and let (p_λ, p_A) be coordinates for P_q defined by

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and let (p_λ, p_A) be coordinates for P_q defined by

$$p_\lambda = \frac{\partial L}{\partial v^\lambda}, \quad p_A = \frac{\partial L}{\partial v^A}, \quad \lambda = 1, \ldots, k, \ A = k+1, \ldots, n,$$

where $v^i = (v^\lambda, v^A)$ are coordinates for $\Delta_Q(q) \subset T_qQ$.

Generalized Legendre Transform

- Notice that the rank of the Hessian is k as

$$\det\left[\frac{\partial^2 L}{\partial v^\lambda \partial v^\mu}\right] \neq 0; \quad \lambda, \mu = 1, \ldots, k \leq n.$$

Define an generalized energy E on $TQ \oplus T^*Q$ by

$$\begin{aligned} E(q^i, v^i, p_i) &= p_i\, v^i - L(q^i, v^i) \\ &= p_\lambda\, v^\lambda + p_A\, v^A - L(q^i, v^\lambda, v^A). \end{aligned}$$

Then, a ***constrained Hamiltonian*** H_P on P can be defined by

$$H_P(q^i, p_\lambda) = \mathrm{stat}_{\,v^i}\, E(q^i, v^i, p_i)\,|\,P.$$

Generalized Hamiltonian

- One can do the ***partial Legendre transform***

$$\mathbb{F}(L|\Delta_Q)\left(q^i,\ v^\lambda\right) = \left(q^i,\ p_\lambda = \frac{\partial L}{\partial v^\lambda}\right)\Bigg|_P$$

and the rest may result in primary constraints.

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$$\phi_A(q^i, p_i) = 0,\ \ A = k+1, \ldots, n.$$

Define the ***generalized Hamiltonian*** H on $TQ \oplus T^*Q$ such that $H \mid P = H_P$, which is locally given by

$$H(q^i, v^A, p_i) = H_P(q^i, p_\lambda) + \phi_A(q^i, p_i)\, v^A,$$

where $v_A,\ A = k{+}1, \ldots, n$ can be regarded as Lagrange multipliers for the primary constraints.

Implicit Hamiltonian Systems

Implicit Hamiltonian Systems

- Let $H : TQ \oplus T^*Q \rightarrow \mathbb{R}$ be the generalized Hamiltonian and the differential of H is locally given by

$$\mathbf{d}H = \left(q^i, v^A, p_i, \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial v^A}, \frac{\partial H}{\partial p_i} \right).$$

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So, restrict $\mathbf{d}H : T(TQ \oplus T^*Q) \to \mathbb{R}$ to TT^*Q and

$$\mathbf{d}H(q, v, p)|_{TT^*Q} = \left(\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i} \right).$$

□ An ***implicit Hamiltonian system*** is defined by (H, Δ_Q, X), which satisfies, for each $(q,p) \in T^*Q$,

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$$\dot{q} = \frac{\partial H}{\partial p} \in \Delta_Q(q), \;\; \dot{p} + \frac{\partial H}{\partial q} \in \Delta_Q^{\circ}(q), \;\; \frac{\partial H}{\partial v^A} = \phi_A(q,p) = 0.$$

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- The ***Hamilton-d'Alembert-Pontryagin principle*** is is given by

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$$= \int_{t_1}^{t_2} \left\{ \left(-\dot{p} - \frac{\partial H}{\partial q} \right) \delta q + \left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \frac{\partial H}{\partial v^A} \delta v^A \right\} dt = 0$$

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Example: Point Vortices

- Start with a degenerate Lagrangian given by

$$L(q^i, v^i) = \langle \alpha_i(q^j),\ v^i \rangle - h(q^i).$$

By computions, we obtain the primary constraints

$$\begin{aligned} \phi_i(q^j, p_j) &= p_i - \frac{\partial L}{\partial v^i} \\ &= p_i - \alpha_i(q^j) = 0, \end{aligned}$$

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- Define an generalized energy E by

$$\begin{aligned}E(q^i, v^i, p_i) &= p_i\, v^1 - L(q^i, v^i) \\ &= (p_i - \alpha_i(q^j))\, v^i + h(q^i)\end{aligned}$$

- The constrained Hamiltonian H_P on P can be defined by

$$
\begin{aligned}
H_P(q^i, p_i) &= \operatorname{stat}_{v^i} E(q^i, v^i, p_i) \mid P \\
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Hence, the generalized Hamiltonian H on $TQ \oplus T^*Q$ can be defined by

$$\begin{aligned} H(q^i, v^i, p_i) &= H_P(q^i, p_i) + \phi_i(q^i, p_i)\, v^i \\ &= h(q^i) + (p_i - \alpha_i(q^j))\, v^i \end{aligned}$$

such that the following relation holds:

$$H \,|\, P = H_P.$$

- The Hamilton-Pontryagin principle in phase space is given (in this case $\Delta_Q = TQ$) by

$$\delta \int_{t_1}^{t_2} \left\{ p_i(t)\, \dot{q}^i(t) - H(q^i, v^A, p_i) \right\}\, dt$$

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$$\dot{q}^i = \frac{\partial H}{\partial p_i} = v^i,$$

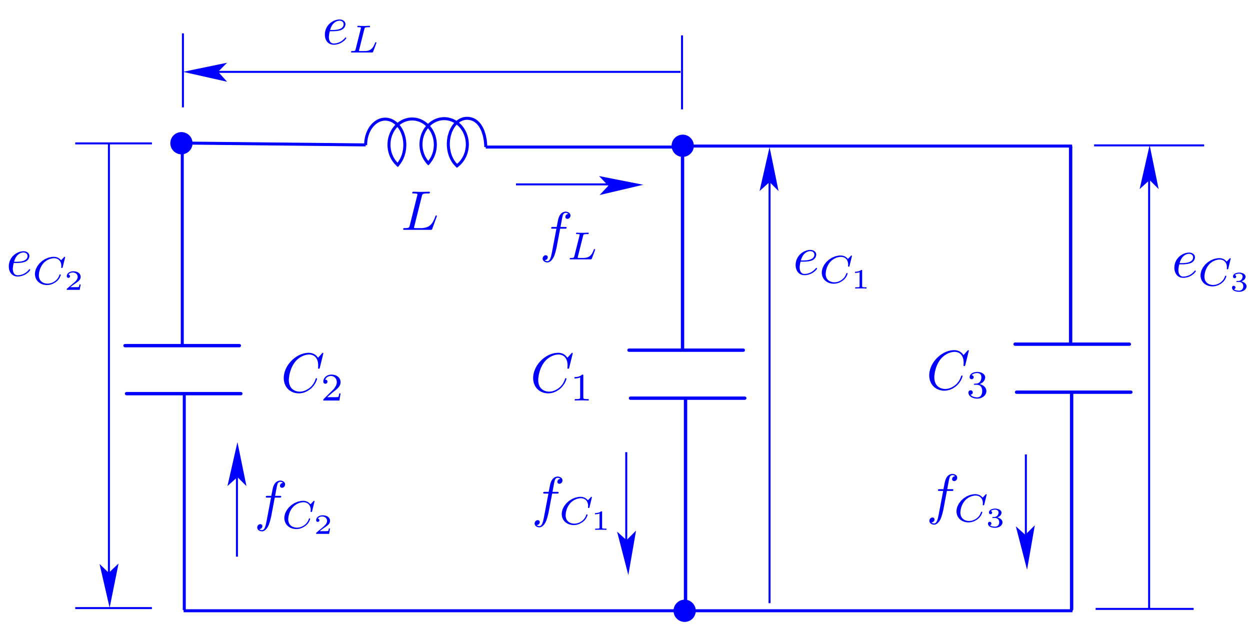
$$\dot{p}^i = -\frac{\partial H}{\partial q^i} = \frac{\partial \alpha_j(q)}{\partial q^i} v^j - \frac{\partial h(q)}{\partial q^i},$$

$$\frac{\partial H}{\partial v^i} = \phi_i(q^j, p_j) = p_i - \alpha_i(q^j) = 0.$$

Example: L-C Circuits

- The generalized energy E on $TW \oplus T^*W$ is given by

$$
\begin{aligned}
E(q^i, f^i, p_i) &= p_i\, f^i - \mathcal{L}(q^i, f^i) \\
&= p_L\, f_L + p_{C_1}\, f_{C_1} + p_{C_2}\, f_{C_2} + p_{C_3}\, f_{C_3} \\
&\quad - \frac{1}{2} L\, (f_L)^2 + \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.
\end{aligned}
$$



Define the constrained Hamiltonian H_P on P by

$$\begin{aligned} H_P(q^i, p_\lambda) &= \operatorname{stat}_{f^i} E(q^i, f^i, p_i) \mid P \\ &= T(q^i, p_\lambda) + V(q^i) \\ &= \frac{1}{2} L^{-1} (p_L)^2 + \frac{1}{2} \frac{(q_{C_1})^2}{C_1} + \frac{1}{2} \frac{(q_{C_2})^2}{C_2} + \frac{1}{2} \frac{(q_{C_3})^2}{C_3}, \end{aligned}$$

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and the primary constraints

$$
\phi_A = 0,\ \ A = 2, 3, 4
$$

are in fact given by

$$
\phi_2 = p_{C_1} = 0,\ \ \phi_3 = p_{C_2} = 0,\ \ \phi_4 = p_{C_3} = 0.
$$

Define the generalized Hamiltonian H on $TW \oplus T^*W$ such that $H \mid P = H_P$, which is locally represented by

$$\begin{aligned} H(q^i, f^A, p_i) &= H_P(q^i, p_\lambda) + \phi_A(q^i, p_i)\, f^A \\ &= \frac{1}{2} L^{-1} (p_L)^2 + \frac{1}{2}\frac{(q_{C_1})^2}{C_1} + \frac{1}{2}\frac{(q_{C_2})^2}{C_2} + \frac{1}{2}\frac{(q_{C_3})^2}{C_3} \\ &\quad + p_{C_1} f_{C_1} + p_{C_2} f_{C_2} + p_{C_3} f_{C_3}, \end{aligned}$$

where we incorporate primary constraints by employing $f^A,\ A = k+1, \ldots, n$ as Lagrange multipliers.

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\mathbf{d}H = \left(q^i, f^A, p_i, \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial f^A}, \frac{\partial H}{\partial p_i} \right).
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We can obtain the primary constraints as

$$\frac{\partial H}{\partial f^A} = \phi_A(q^i, p_i) = p_A = 0, \quad A = 2, 3, 4.$$

The restriction of $\mathbf{d}H : T(TW \oplus T^*W) \to \mathbb{R}$ to TT^*W is locally denoted by

$$\begin{aligned}
\mathbf{d}H(q^i, v^A, p_i)|_{TT^*W} &= \left(\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i}\right) \\
&= \left(0, \frac{q_{C_1}}{C_1}, \frac{q_{C_2}}{C_2}, \frac{q_{C_3}}{C_3}, p_L, p_{C_1}, p_{C_2}, p_{C_3}\right) \\
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- The vector field X on T^*W, defined at points in P, is locally represented by

$$X\left(q_L, q_{C_1}, q_{C_2}, q_{C_3}, p_L, 0, 0, 0\right) = \left(\dot{q}_L, \dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}, \dot{p}_L, 0, 0, 0\right),$$

and the condition of an implicit Hamiltonian system (H, Δ, X) is satisfied such that for each $(q, p) \in T^*W$,

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$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p^i} \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a(q) \end{pmatrix},$$

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- Recall the generalized energy $E : TQ \oplus T^*Q \to \mathbb{R}$ is defined by

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So, restrict $\mathbf{d}E : T(TQ \oplus T^*Q) \to \mathbb{R}$ to TT^*Q and

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□ The implicit Lagrangian system (L, Δ_Q, X) that satsfies the condition

$$(X, \mathfrak{D}L) \in D_{\Delta_Q}$$

can be restated by, for each $(q, p) \in T^*Q$,

$$(X(q, p), \mathbf{d}E(q, v, p)|_{TT^*Q}) \in D_{\Delta_Q}(q, p).$$

Passage from ILS to IHS

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Let's go to the Hamiltonian side!

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But, unfortunately, you can ***never come back to the Lagrangian side*** *from the Hamiltonian side in the degenerate cases! It's a* ***one way*** *passage!*

Concluding Remarks

- We have showed the *link between implicit Lagrangian and Hamiltonian systems* in the case that a given Lagrangian is degenerate.
- We have developed a *generalized Legendre transform* for degenerate Lagrangians and also developed a *generalized Hamiltonian* on the Pontryagin bundle, by which we can incorporates *primary constraints* into the variational as well as into the Dirac context.
- We have developed implicit Hamiltonian systems for degenerate cases in the context of Dirac structures as well as in the context of the *Hamilton-Lagrange-Pontryagin principle* together with some examples.