Implicit Analytic Differential Equations

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CONTENTS

1.- Basic examples

- $\phi(x, \dot{x}) = 0$, general form of an implicit differential equation (IDE).
- $\phi : TM \to F$, M manifold, F vector space.
- ϕ belongs to a certain class of functions, for instance, smooth, analytic, etc.
- Nonholonomic systems

$$\delta \int L(x, \dot{x}) dt = 0, \quad \varphi(x, \delta x) = 0$$

$$\varphi(x, \dot{x}) = 0.$$

- $(x, \dot{x}) = (x, v)$ then the previous system of equations becomes an IDE, $\psi(x, v, \dot{x}, \dot{v}) = 0$.
- Reduced versions of nonholonomic systems also lead to IDE, sometimes easier. Question: find a good systematic way of writing an equivalent ODE, at least for a certain class of nonholonomic systems.
- Singular Lagrangian systems
- $\delta \int L(x, \dot{x}) dt = 0$, no restriction on δx, with the same substitution $(x, \dot{x}) = (x, v)$ leads to Euler-Lagrange equation, which are IDE

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j + B_i(x, v) = 0.$$

- Hamilton-Poincaré equations.

$$\delta \int (p\dot{q} - (pv - L(q, v)))\, dt = 0,$$

gives the equations

$$\omega(q, v, p)\,((\dot{q}, \dot{v}, \dot{p}), \quad) = dH(q, v, p),$$

where $H(q, v, p) = pv - L(q, v)$ and $\omega = dq \wedge dp$ is a presymplectic form.

- More generally, an equation similar to this on a given presymplectic manifold, say $\omega(x)(\dot{x}, \quad) = \alpha$, where α is a 1-form (even in infinite dimensions) is the starting point of the theory of Gotay-Nester, which generalizes the Dirac-Bergman theory of constraints.

- An even more general class of examples: instead of a presymplectic structure one can write equations in terms of a given Dirac structure,which in general also lead to IDE.

- Control systems of the type

$$\dot{x} = f(x, y)$$
$$0 = g(x, y)$$

are obviously another source of examples of IDE

2.- The algorithm

- M manifold of dimension d, (a, f) given IDE with domain M and range F. The algorithm is designed to transform this IDE into an equivalent IDE, say

$$\tilde{a}_2(y)\dot{y} = \tilde{f}_2(y)$$

on an analytic manifold $\tilde{M}_2$, which is an IDE of locally constant rank. This involves a desingularization process

- The decomposition $M = M_0 \cup M_1 \cup M_2$. First, let us assume that M is a connected manifold of dimension d. For $i = 0, 1, \ldots,$ let

$$S_i(M) = \{x \in M \mid \operatorname{rank} a(x) \leq i\}$$

$S_i(M)$ is clearly a closed analytic subset of M, defined by analytic equations, for $i = 0, 1, \ldots$.

- Also, for $i = 0, 1, ...,$ let $L_i(M) \subseteq S_i(M)$ be defined by

$$L_i(M) = \{x \in S_i(M) \mid \operatorname{rank}[a(x), f(x)] \leq i\}$$

Each $L_i(M)$ is a closed analytic subset of M defined by analytic equations.

- Let

$$S_{k_1}(M) \subset S_{k_2}(M) \subset \ldots \subset S_{k_r}(M)$$

be the distinct nonempty $S_i(M)$. We observe that $S_{k_r}(M) \equiv M$. Consider the corresponding inclusions

$$L_{k_1}(M) \subseteq L_{k_2}(M) \subseteq \ldots \subseteq L_{k_r}(M).$$

- $\operatorname{rank} a(x) = \operatorname{rank}[a(x), f(x)] = k_j$ for each $x \in L_{k_j}(M) - S_{k_{j-1}}(M)$, $j = 1, \ldots, r$. The LAS associated to (a, f) has solution for each $x \in L_{k_j}(M) - S_{k_{j-1}}(M)$, $j = 1, \ldots, r$, where we have written $S_{k_0} = \emptyset$, to uniformize the notation.

- We remark the following useful facts: the set $L_{k_j}(M) - S_{k_j-1}(M)$ may be empty, for some $j = 1, \ldots, r$; we have $\dim S_{k_{r-1}}(M) < \dim M$; if $\dim(L_{k_r}(M)) = d$, then $L_{k_r}(M) = M$.

- Now let M be a manifold of dimension d and assume that

$$M_m = \bigcup_j W_j$$

is the union of the connected components of M of maximal dimension d.

- We will consider the following pairwise disjoint conditions for a given $W_j \subseteq M_m$,
  (a) $L_{k_r}(W_j) = \emptyset$
  (b) $L_{k_r}(W_j) \neq \emptyset$ and $\dim L_{k_r}(W_j) < d$
  (c) $L_{k_r}(W_j) \neq \emptyset$ and $\dim L_{k_r}(W_j) = d$.
- We now define the following pairwise disjoint subsets of M.

$$M_0 = (M - M_m) \cup \bigcup_b L_{k_r}(W_j) \cup \bigcup_c S_{k_{r-1}}(W_j)$$

$$M_1 = \bigcup_a W_j \cup \bigcup_b (W_j - L_{k_r}(W_j))$$

$$M_2 = \bigcup_c \left(W_j - S_{k_{r-1}}(W_j)\right).$$

- We have the following assertions, whose proof is easy: each subset $L_{k_r}(W_j) \subseteq W_j$, and each subset $S_{k_{r-1}}(W_j) \subseteq W_j$, is a closed analytic subset of W_j defined by analytic equations on W_j. In consequence, $W_j - L_{k_r}(W_j)$, $W_j - S_{k_{r-1}}(W_j)$ are open submanifolds of W_j.
- The manifold M is the disjoint union

$$M = M_0 \cup M_1 \cup M_2.$$

  M_1 and M_2 are open submanifolds of M. and M_0 is a union of subsets defined by analytic equations on each W_j, union $M - M_m$, and we have that $\dim M_0 < d$.

- We have that the LAS associated to (a, f) has no solution for $x \in M_1$. On the other hand, it has solution for all $x \in M_2$, moreover, $(a, f)|M_2$, *is an IDE of locally constant rank.*

- It remains to see what happens with the system restricted to M_0. The idea here is to desingularize each closed analytic subset $L_{k_r}(W_j) \subseteq W_j$, and $S_{k_{r-1}}(W_j) \subseteq W_j$. By forming the disjoint union of those desingularizations and $M - M_m$ one obtains a desingularization of M_0 say

$$\pi_0 : M^1 \to M, \text{ where } \pi_0(M^1) = M_0.$$

- Then $(a, f)|M_0$ can be naturally *lifted* to an IDE $(a_1, f_1) = \pi_0^* ((a, f)|M_0)$ on M^1 as follows

$$a_1(y)\dot{y} = a(\pi_0(y))T_y\pi_0(y, \dot{y})$$
$$f_1(y) = f(\pi_0(y)).$$

- Note that M^1 is a manifold of dimension $\dim M^1 = \dim M_0 < d$.

To desingularize (a, f) in a finite number of steps we repeat the process for the IDE (a_1, f_1) with domain M^1 and range F, as we did before with (a, f).

- We obtain a decomposition

$$M^1 = M_0^1 \cup M_1^1 \cup M_2^1 .$$

We know that there is no solution to the LAS system

$$a_1(y)\dot{y} = f_1(y)$$

for $y \in M_1^1$. We also know that there is solution to the same LAS system for $y \in M_2^1$, moreover, $(a_1, f_1)|M_2^1$ is an IDE of locally constant rank. Now we desingularize M_0^1

$$\pi_1 : M^2 \to M^1,\ \pi_1(M^2) = M_0^1$$

and repeat the process. It is clear that we obtain a finite sequence of manifolds and maps

$$M^q \overset{\pi_{q-1}}{\to} M^{q-1} \overset{\pi_{q-2}}{\to} \dots \overset{\pi_1}{\to} M^1 \overset{\pi_0}{\to} M,$$

where $\pi_0(M^1) = M_0$, $\pi_1(M^2) = M_0^1$, and in general $\pi_i(M^{i+1}) = M_0^i$, for $i = 0, ..., q-1$, where we have written $M^0 \equiv M$ to unify the notation.

- We have obtained a finite recursive procedure that reduces the problem to a finite number of IDE of locally constant rank, namely, the IDE of locally constant rank $(a_i, f_i)|M_2^i$, for $i = 0, 1, ..., q$, where we have written $(a_0, f_0) = (a, f)$, to unify the notation.

- We will call this a *desingularization process* and the sequence of maps π_i and IDE (a_{i+1}, f_{i+1}), $i = 0, ..., q-1$ a *desingularization* of (a, f).

- The collection of IDE $(a_k, f_k)|M_2^k$, $k = 0, ..., q$, defines a single IDE $(\tilde{a}_2, \tilde{f}_2)$ of locally constant rank in the disjoint union $\tilde{M}_2 = \bigsqcup_{k=0}^{q} M_2^k$. We have a natural projection $\tilde{\pi}_2 : \tilde{M}_2 \to M$. This IDE $(\tilde{a}_2, \tilde{f}_2)$ with domain $\tilde{M}_2$ and range F is called the *desingularizing IDE*.

- Are the given system and the desingularized system *equivalent?* It is easy to see that any solution of the desingularized sytems is projected into a solution of the original syste. The converse is not easy to prove.

- The main difficulty for proving the converse comes from the following problem. Let $f : X \to Y$ be given and let $y(t)$ be a given curve in Y. Question: Is it true that there exists a curve $x(t)$ in X such that $y(t) = f(x(t))$?.

- I do not know the answer to this question in the category of C^∞ manifolds and maps. But there is a certain affirmativr answer in the category of real analytic manifolds and maps. This is possible thanks to the theory of semianalytic and subanalytic sets, developed by Lojasiewicz Hardt, Hironaka, Bierstone, Milman, Sussmann and many others. We will show later in this talk the main lines of the proof of the equivalenc in the analytic case. This case is important because of the many problems in mechanics that belong to it.
- Before we do that we will see an example from nonholonomic mechanics.

3.- The symmetric ball rolling without sliding or spinning.

- Kinematics. Let an ortonormal system fixed in the space, say (e_1, e_2, e_3), $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, then we have a basis moving with the body, (Ae_1, Ae_2, Ae_3), where $A = A(t)$. We introduce the variable $z \in S^2$, given by $z = Ae_3$. The spatial angular velocity ω can be written $\omega = v_0 z + z \times \dot{z}$, so $v_0 = \langle \omega, z \rangle$ is the component of ω along z. To the usual nonsliding condition $\omega \times re_3 = \dot{x}$ for the rigid rolling sphere we must add the extra condition that the vertical component of the spatial angular velocity is 0, that is, $\omega_3 = 0$.

- Dynamics. We are going to assume that the center of mass coincides with the center of the sphere and that the principal axis of inertia are (Ae_1, Ae_2, Ae_3). The three principal moments of inertia of the sphere are I_1, I_2, I_3, and we are going to assume that $I_1 = I_2$. We introduce the adimensional quantities $\alpha = I_3/I_1$ and $\beta = Mr^2/I_1$, where M is the mass of the sphere.

- The Lagrangian of the system is given by the kinetic energy,

$$\frac{1}{2}I_1\dot{z}^2 + \frac{1}{2}I_3 v_0^2 + \frac{1}{2}M\dot{x}^2$$

where $\dot{x}$ is the velocity of the center of the sphere. The nonholonomic constraint is given by $\dot{x} = \omega \times re_3$ and $\omega_3 = 0$, and using this we can conclude that the kinetic energy of the actual motion of the symmetric sphere is given by

$$E = \frac{1}{2}(I_1 + Mr^2)\dot{z}^2 + \frac{1}{2}(I_3 + Mr^2)v_0^2.$$

- The addition of the extra condition $\omega_3 = 0$ introduces an extra singularity in the reduced system, which is an IDE. We will apply our desingularization procedure to obtain a single equivalent differential equation describing the system. The desingularized manifold containing the essential dynamics is diffeomorphic to $S^2 \times S^1$. Integrability by quadratures appears in a natural way.

- The IDE for the symmetric elastic sphere. As a result of reduction by the symmetry techniques, in this case reduction by the subgroup $SO(2) \times \mathbb{R}^2$ we obtain the following system of Lagrange-D'Alembert-Poincaré equations, which is an IDE,

$$\begin{aligned} (\alpha+\beta)(z \times e_3)\dot{v}_0 + (1+\beta) < z, e_3 > \nabla_{\dot{z}}\dot{z} - \\ (\alpha+\beta)v_0 < z, e_3 > (z \times \dot{z}) &= 0 \\ v_0 < z, e_3 > + < z \times \dot{z}, e_3 > &= 0. \end{aligned}$$

Here ∇ represents the Levi-Civita connection on S^2 with respect to the standard metric. The previous Lagrange-D'Alembert-Poincaré equations are derived under the assumption $z_3 \neq 0$ because the so called *dimension assumption* is not satisfied for the whole manifold S^2. Nevertheless, by continuity, these are also satisfied by the motion of the rolling ball for $z_3 = 0$. One can also check that these equations are consistent with balance of momentum (Newton's law).

- Preservation of energy. By working with the previous equations one gets

$$0 = \frac{d}{dt}\left((1+\beta)\dot{z}^2 + (\alpha+\beta)v_0^2\right),$$

from which one deduces

$$2\epsilon = (1+\beta)\dot{z}^2 + (\alpha+\beta)v_0^2,$$

where ϵ represents the normalized energy. This equation represents conservation of energy, as one can check more directly by looking at the expression of the kinetic energy E given before. We shall assume from now on that $\epsilon > 0$, otherwise the motion is trivial.

- We have the following equations to be satisfied for the symmetric elastic sphere in variables (z, u) where $\dot{z} = v$ and $v \times z = u$, so the variable v_0 does not appears,

$$\begin{aligned} (1+\beta) < z, e_3 >< \dot{u}, e_3 \times z > & \\ +(\alpha+\beta) < u, e_3 >^2 &= 0 \\ (1+\beta) < z, e_3 >^2 u^2 + (\alpha+\beta) < u, e_3 >^2 & \\ -2\epsilon < z, e_3 >^2 &= 0, \end{aligned}$$

- The IDE for the symmetric elastic sphere in the standard form. Considering that $z^2 = 1$ and $\dot{z} = z \times u$, we must have $< z, u >= 0$. Using what was said before we can write the system of equations for the symmetric elastic sphere in variables $(z, u, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ as follows,

$$
\begin{aligned}
\dot{z}_1 &= z_2 u_3 - z_3 u_2 \\
\dot{z}_2 &= z_3 u_1 - z_1 u_3 \\
\dot{z}_3 &= z_1 u_2 - z_2 u_1 \\
0 &= (1+\beta) z_3(-z_2 \dot{u}_1 + z_1 \dot{u}_2) + (\alpha+\beta) u_3^2 \\
0 &= (1+\beta) z_3^2 (u_1^2 + u_2^2 + u_3^2) + (\alpha+\beta) u_3^2 - 2\epsilon z_3^2 \\
0 &= z_1^2 + z_2^2 + z_3^2 - 1 \\
0 &= z_1 u_1 + z_2 u_2 + z_3 u_3 \\
0 &= 2\epsilon - (1+\beta) u^2 - (\alpha+\beta) v_0^2.
\end{aligned}
$$

This system can be written in the form

$$a(X)\dot{X} = f(X),$$

with $X = (z, u, v_0)$, where

-

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+\beta)z_2 z_3 & (1+\beta)z_1 z_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f = \begin{bmatrix} z_2 u_3 - z_3 u_2 \\ z_3 u_1 - z_1 u_3 \\ z_1 u_2 - z_2 u_1 \\ -(\alpha+\beta)u_3^2 \\ (1+\beta)z_3^2(u_1^2+u_2^2+u_3^2) + (\alpha+\beta)u_3^2 - 2\epsilon z_3^2 \\ z_1^2 + z_2^2 + z_3^2 - 1 \\ z_1 u_1 + z_2 u_2 + z_3 u_3 \\ 2\epsilon - (1+\beta)u^2 - (\alpha+\beta)v_0^2 \end{bmatrix}.$$

- Application of the algorithm.We will work on the manifold $M=\mathbb{R}^7$, where $(z_1,z_2,z_3,u_1,u_2,u_3,v_0)\in\mathbb{R}^7$ are independent variables. We can easily see that $k_r=4$, $S_4(M)=M$, $L_4(M)=M_0$, $M_1=M-L_4(M)$, $M_2=\emptyset$. Now we shall describe M_0 by equations. Let

  $$\varphi_1=-(1+\beta)z_2z_3$$
  $$\varphi_2=(1+\beta)z_1z_3$$
  $$\nu_1=(1+\beta)z_3^2(u_1^2+u_2^2+u_3^2)+(\alpha+\beta)u_3^2-2\epsilon z_3^2$$
  $$\nu_2=z_1^2+z_2^2+z_3^2-1$$
  $$\nu_3=z_1u_1+z_2u_2+z_3u_3$$
  $$\nu_4=2\epsilon-(1+\beta)u^2-(\alpha+\beta)v_0^2.$$

  As we know $M_0=L_4(M)$ is given by the condition that $\operatorname{rank}[a,f]\leq 4$. Let

  $$\begin{aligned}M_{0a}&=\{\varphi_1=0,\varphi_2=0\}\\&=\{z_3=0\}\cup\{z_1=0,z_2=0\}\\M_{0b}&=\{\nu_1=0,\nu_2=0,\nu_3=0,\nu_4=0\}.\end{aligned}$$

  Then we can easily see that $M_0=M_{0a}\cup M_{0b}$. The desingularization M^1 of M_0 will be the disjoint union of the desingularizations of M_{0a} and M_{0b}.

- The desingularization M_a^1 of M_{0a} can be described by $M_a^1 \equiv \{z_3 = 0\} \bigsqcup \{z_1 = 0, z_2 = 0\}$, where $\bigsqcup$ means *disjoint union* and the projection π_0 is the identity on each disjoint piece of M_a^1. One can see using (1) - (1) that the lifted system $(a_1, f_1)|\{z_3 = 0\}$ satisfies $z_3 = 0$, $u_3 = 0$, $z_1^2 + z_2^2 = 1$, which implies $\dot{z} = 0$, and also, since $u = \dot{z} \times z$, that $u = 0$. This describes the motion completely. It consists of the rolling of the sphere with $z(t) = (z_{10}, z_{20}, 0)$ fixed and the z component of the angular velocity v_0 satisfies $2\epsilon = (\alpha + \beta)(v_0)^2$. The lifted system $(a_1, f_1)|\{z_1 = 0, z_2 = 0\}$ satisfies $z_1 = 0$, $z_2 = 0$, $z_3 = \pm 1$, therefore $\dot{z} = 0$, and then $u = 0$, which contradicts equation $\nu_1 = 0$, because we have assumed $\epsilon > 0$. So there is no motion, that is, no solution, for the system $(a_1, f_1)|\{z_1 = 0, z_2 = 0\}$.
- Now we will desingularize M_{0b}. We are going to see that M_{0b} is in fact a nonsingular manifold. More precisely, we will define the desingularizing manifold M_b^1 by equations in the variables (z, u, v_0), with $v_0 z_3 = u_3$).

- For simplicity, we call $\mu = 2\epsilon/(1+\beta) > 0$ and $\lambda = (\alpha+\beta)/(1+\beta) > 0$, from now on. Then we have the following equations defining the nonsingular manifold M_b^1,

$$
\begin{aligned}
0 &= u_3 - v_0 z_3 \\
0 &= u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu \\
0 &= z_1^2 + z_2^2 + z_3^2 - 1 \\
0 &= z_1 u_1 + z_2 u_2 + z_3 u_3.
\end{aligned}
$$

The proof of this is straightforward. The map $\pi_0 : M_b^1 \to M$ is then given by the restriction of the identity $(z, u, v_0) \to (z, u, v_0)$ to M_b^1. Then, the original system would be equivalent to the following system

$$
\begin{aligned}
\dot{z}_1 &= z_2 u_3 - z_3 u_2 \\
\dot{z}_2 &= z_3 u_1 - z_1 u_3 \\
\dot{z}_3 &= z_1 u_2 - z_2 u_1 \\
z_2 \dot{u}_1 - z_1 \dot{u}_2 &= \lambda v_0 u_3 \\
0 &= u_3 - v_0 z_3 \\
0 &= u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu \\
0 &= z_1^2 + z_2^2 + z_3^2 - 1 \\
0 &= z_1 u_1 + z_2 u_2 + z_3 u_3.
\end{aligned}
$$

- More precisely, if we define

$$\tilde{a}(z,u,v_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 & -z_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{f}(z,u,v_0) = \begin{bmatrix} z_2u_3 - z_3u_2 \\ z_3u_1 - z_1u_3 \\ z_1u_2 - z_2u_1 \\ \lambda v_0 u_3 \\ u_3 - v_0 z_3 \\ u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu \\ z_1^2 + z_2^2 + z_3^2 - 1 \\ z_1u_1 + z_2u_2 + z_3u_3 \end{bmatrix},$$

we see that the previous system is in the form $\tilde{a}(y)\dot{y} = \tilde{f}(y)$, with $y = (z,\ u,\ v_0)$, so it is an IDE in standard form with domain $\mathbb{R}^7$ and range $\mathbb{R}^8$, and our IDE in M_b^1 is given by the restriction $(a_1, f_1) = (\tilde{a}, \tilde{f})|M_b^1$.

- In order to continue with the algorithm, we shall find explicitly the lifted system (a_1, f_1). By differentiating the first seven of the previous equations, eliminating the redundant equation $z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 = 0$, also realizing appropriate linear operations in the range space, we have the following system with domain $\mathbb{R}^7$ and range $\mathbb{R}^{11}$, which is also equivalent to our system (a_1, f_1),

$$
\begin{aligned}
\dot{z}_1 &= z_2u_3 - z_3u_2 \\
\dot{z}_2 &= z_3u_1 - z_1u_3 \\
\dot{z}_3 &= z_1u_2 - z_2u_1 \\
z_2\dot{u}_1 - z_1\dot{u}_2 &= \lambda v_0 u_3 \\
z_1\dot{u}_1 + z_2\dot{u}_2 + z_3\dot{u}_3 &= 0 \\
u_1\dot{u}_1 + u_2\dot{u}_2 + u_3\dot{u}_3 + \lambda v_0\dot{v}_0 &= 0 \\
\dot{u}_3 - z_3\dot{v}_0 &= v_0z_1u_2 - v_0z_2u_1 \\
0 &= u_3 - v_0z_3 \\
0 &= u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 \\
&\quad -\mu \\
0 &= z_1^2 + z_2^2 + z_3^2 - 1 \\
0 &= z_1u_1 + z_2u_2 + z_3u_3.
\end{aligned}
$$

This system is still not completely desingularized.

- One can check by direct calculation that it can be desingularized in two more iterations of the algorithm. However, in this example there is an interesting alternative to find the solutions, which starts with a precise description of the manifold M_b^1. We prefer this alternative because having an identification of M_b^1 also helps to understand the dynamics in a direct way, as we will see soon.

- Solving the equations

  Identification of M_b^1. The manifold M_b^1 is given by the equations in the space of the variables $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$, as we have seen before. These equation tells us that u is a vector tangent to the 2-sphere S^2, given by $z^2 - 1 = 0$. Heuristically, for each $z \in S^2$ we consider the 3 dimensional space $T_z S^2 \times R_z$, where R_z represents a line normal to the sphere at $z \in S^2$, so the collection of all R_z is a trivial real line vector bundle with base S^2.

- One of the equation is a plane containing the origin $0 = 0_z$ since z_3 is fixed once z is fixed. Another equation gives an ellipsoid. The intersection of the plane with the ellipsoid is an ellipse. Therefore M_b^1 must be some fiber bundle with fiber S^1 and base S^2. Using all this and some imagination we can see that it is, in fact, the trivial bundle $S^2 \times S^1$, moreover, we have the following parametrization of M_b^1 in variables $(\theta,\ \varphi,\ \psi)$. In any case, this assertion can be easily checked after some straightforward calculations.

- $$\begin{aligned} z_1 &= \sin\theta\cos\varphi \\ z_2 &= \sin\theta\sin\varphi \\ z_3 &= \cos\theta \\ u_1 &= -a\cos(\varphi-\psi)\cos^2\theta\cos\varphi - b\sin(\varphi-\psi)\sin\varphi \\ u_2 &= -a\cos(\varphi-\psi)\cos^2\theta\sin\varphi + b\sin(\varphi-\psi)\cos\varphi \\ u_3 &= a\cos(\varphi-\psi)\cos\theta\sin\theta \\ v_0 &= a\cos(\varphi-\psi)\sin\theta, \end{aligned}$$

  where
  $$a = \sqrt{\frac{\mu}{\lambda\sin^2\theta + \cos^2\theta}}, \;\; b = \sqrt{\mu}'.$$
  In other words, by some straightforward calculations we can check that $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$ in coordinates $(\theta,\, \varphi,\, \psi)$ satisfies the equations.

- We can see that the previous system of equations define a diffeomorphism $f : S^2 \times S^1 \to M_b^1$, $f(z, (\cos\psi, \sin\psi)) = (z, u, v_0)$, which gives the desired identification of M_b^1. This takes some long but straightforward calculations.

- The differential equation for the symmetric elastic sphere in variables (θ, φ, ψ). We get the equations in coordinates (θ, φ, ψ) as follows

$$\cos\theta\cos\varphi\dot\theta - \sin\theta\sin\varphi\dot\varphi = a\cos\theta\sin\varphi\cos(\varphi-\psi) - b\cos\theta\cos\varphi\sin(\varphi-\psi)$$

$$\cos\theta\sin\varphi\dot\theta + \sin\theta\cos\varphi\dot\varphi = -a\cos\theta\cos\varphi\cos(\varphi-\psi) - b\cos\theta\sin\varphi\sin(\varphi-\psi)$$

$$-\sin\theta\dot\theta = b\sin\theta\sin(\varphi-\psi)$$

$$a\sin\theta\cos^2\theta\cos(\varphi-\psi)\dot\varphi-$$

$$-b\sin\theta\cos(\varphi-\psi)(\dot\varphi-\dot\psi) = \lambda a^2\cos^2(\varphi-\psi)\sin^2\theta\cos\theta$$

If $\sin\theta \neq 0$ the system becomes

$$\begin{aligned}\dot\theta &= -b\sin(\varphi-\psi)\\ \dot\varphi &= -a\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi)\\ \dot\psi &= a\cos(\varphi-\psi)\frac{\cos\theta}{\sin\theta}\left(\frac{b}{a}-1\right),\end{aligned}$$

or equivalently,

$$\begin{aligned}\dot\theta &= -b\sin(\varphi-\psi)\\ \dot\varphi &= -a\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi)\\ \dot\psi &= (b-a)\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi)\end{aligned}$$

- It can be easily seen that this system can be integrated by quadratures. For instance, if we call $w = \varphi - \psi$, we can write (1)-(1) as a planar system in coordinates (θ, w),

$$\begin{aligned} \dot{\theta} &= -b \sin w \\ \dot{w} &= -b \frac{\cos\theta}{\sin\theta} \cos w, \end{aligned}$$

which in turn leads to the separable equation

$$\frac{d\theta}{dw} = \tan\theta \tan w.$$

Of course the system is still an analytic IDE on $S^2 \times S^1$ which is of constant rank for $\sin\theta \neq 0$, and the rank changes for $\sin\theta = 0$. So we should continue the desingularization process. Instead, we observe that the only solution with some initial condition compatible with the system and involving the condition $\sin\theta = 0$, that is, $(z_{10}, z_{20}, z_{30}, u_{10}, u_{20}, u_{30}, v_{00}) = (0, 0, \pm 1, u_{10}, u_{20}, 0, 0)$, is a uniform circular motion of z on a vertical plane perpendicular to the constant vector $(u_1(t), u_2(t), u_3(t)) = (u_{10}, u_{20}, 0)$, while $v_0(t) = 0$. This is also consistent with physical reasoning.

4. Equivalence between a given IDE and its desingularization

The following assertions establish the equivalence between a given IDE (a, f) and its desingularization.

(a) Let $y(t)$, $t \in [t_0, t_1)$ (respectively, $t \in (t_0, t_1],$) be an as-solution to (a_k, f_k) in M^k, $k = 1, ..., q$. Then $x(t) = \pi_{k-1}(y(t))$, $t \in [t_0, t_2]$ (respectively, $t \in [t_2, t_1]$) is an as-solution to (a_{k-1}, f_{k-1}) in M^{k-1}, for each $t_2 \in (t_0, t_1)$.

(b) If $x(t)$, $t \in [t_0, t_1)$ (respectively, $t \in (t_0, t_1]$) is an as-solution to (a_{k-1}, f_{k-1}) in M^{k-1} such that $x(t) \in M_0^{k-1}$, $t \in [t_0, t_1)$ (respectively, $t \in (t_0, t_1]$), $k = 1, ..., q$ then there exists $t_2 \in (t_0, t_1)$ and a lifted as-solution $y(t)$, $t \in [t_0, t_2]$ (respectively, $t \in [t_2, t_1]$) of $x|[t_0, t_2]$ (respectively, $x|[t_2, t_1]$) to (a_k, f_k) in M^k, in particular, $x(t) = \pi_{k-1}(y(t))$, $t \in [t_0, t_2]$ (respectively, $t \in [t_2, t_1]$.)

5. Future work.

- The well known *constraint algorithm* of Gotay and Nester gives a global geometric version of the Dirac-Bergman theory of constraints, for infinite dimensional presymplectic manifolds.

- We will briefly show how to deal with the singular cases, in finite dimensions.
- Let (M, ω) be a presymplectic manifold, where the manifold M and the presymplectic form ω are analytic. Let α be a closed analytic 1-form on M. Then the problem is to find solutions to the differential equation

$$\mathrm{i}_{\dot{x}}\,\omega = \alpha.$$

- To write this IDE with the notation used in the present paper let us assume, for simplicity, that M is an open subset of the vector space E, then $TM = M \times E$ and $T^*M = M \times E^*$.

- Then we can write the equation in the form

$$a(x)\dot{x} = f(x),$$

where $a : TM \to F$ by simply taking $F \equiv E^*$, $f(x) \equiv \alpha(x)$ and $a(x)\dot{x} \equiv \omega(x)(\dot{x},\ )$.

- The Gotay-Nester algorithm involves finding a sequence of constraints submanifolds $M \supseteq M_1 \supseteq, ... \supseteq M_q$. M_q is called the *final constraint submanifold* and the dynamics of the system takes place on this submanifold.

- It is *assumed as part of the algorithm that each M_i is a submanifold.* If we apply our algorithm, this assumption is no longer necessary, and the dynamics is recovered as the dynamics of the *system of locally constant rank* $(\tilde{a}_2, \tilde{f}_2)$ *in the analytic manifold* $\tilde{M}_2$. Thus $\tilde{M}_2$ will be the final constraint manifoldd. In this sense, our algorithm represents a generalization of the Gotay-Nester algorithm.

THANK YOU FOR YOUR ATTENTION

GRACIAS POR SU ATENCIÓN