Hopf Bifrucation

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Consider the planar system

$$\dot{x} = -y + x(\mu - x^2 - y^2)$$
$$\dot{y} = x + y(\mu - x^2 - y^2)$$

The only critical point is at the origin and

$$D\mathbf{f}(\mathbf{0}, \mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

The origin is a stable or an unstable focus of this nonlinear system if $\mu < 0$ or if $\mu > 0$ respectively. To uncover the structure at $\mu = 0$, we rewrite the system in polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$, $\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y})$ and $\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x})$ to find that

$$\dot{r} = r(\mu - r^2)$$
$$\dot{\theta} = 1$$

We see that at $\mu = 0$ the origin is a stable focus and for $\mu > 0$ there is a stable invariant orbit

$$\Gamma_\mu : \gamma_\mu(t) = \sqrt{\mu}(\cos t, \sin t)^T$$

The curves Γ_μ represent a one-parameter family of invariant orbits of this system.

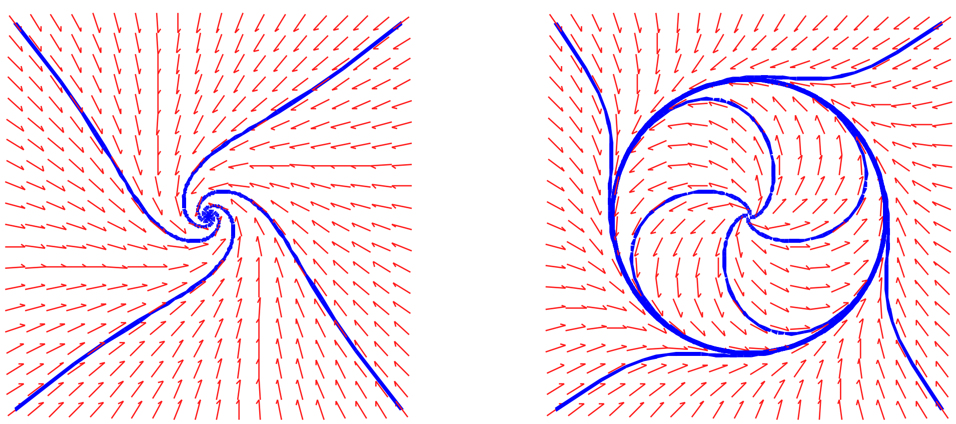

Figure 1: *Left: Phase portrait with $\mu < 0$ illustrating a stable node at the origin. Right: A Hopf bifurcation occurs when $\mu > 0$, and an invariant stable orbit is born with radius $\sqrt{\mu}$ as depicted above.*

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