density of periodic points and transitivity are enough to guarantee sensitive dependence. See [8]. We will delve more deeply into chaotic behavior of discrete systems in the next chapter.

14.6 Exploration: The Rössler Attractor

In this exploration, we investigate a three-dimensional system similar in many respects to the Lorenz system. The Rössler system [37] is given by

$$
\begin{aligned}
x' &= -y - z \\
y' &= x + ay \\
z' &= b + z(x - c)
\end{aligned}
$$

where a, b, and c are real parameters. For simplicity, we will restrict attention to the case where $a = 1/4$, $b = 1$, and c ranges from 0 to 7.

As with the Lorenz system, it is difficult to prove specific results about this system, so much of this exploration will center on numerical experimentation and the construction of a model.

1. First find all equilibrium points for this system.
2. Describe the bifurcation that occurs at $c = 1$.
3. Investigate numerically the behavior of this system as c increases. What bifurcations do you observe?
4. In Figure 14.12 we have plotted a single solution for $c = 5.5$. Compute other solutions for this parameter value, and display the results from other

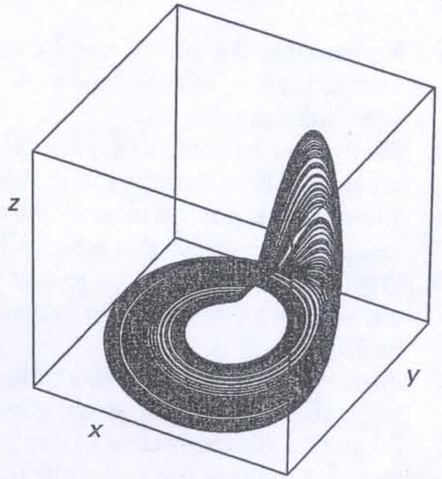

Figure 14.12 The Rössler attractor.

viewpoints in $\mathbb{R}^3$. What conjectures do you make about the behavior of this system?

5. Using techniques described in this chapter, devise a geometric model that mimics the behavior of the Rössler system for this parameter value.
6. Construct a model mapping on a two-dimensional region whose dynamics might explain the behavior observed in this system.
7. As in the Lorenz system, describe a possible way to reduce this function to a mapping on an interval.
8. Give an explicit formula for this one-dimensional model mapping. What can you say about the chaotic behavior of your model?
9. What other bifurcations do you observe in the Rössler system as c rises above 5.5?

EXERCISES

1. Consider the system

$$x' = -3x$$
$$y' = 2y$$
$$z' = -z.$$

Recall from Section 14.4 that there is a function $h\colon \mathcal{R}_1 \to \mathcal{R}_2$ where $\mathcal{R}_1$ is given by $|x| \le 1$, $0 < y \le \epsilon < 1$ and $z = 1$, and $\mathcal{R}_2$ is given by $|x| \le 1$, $0 < z \le 1$, and $y = 1$. Show that h is given by

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} xy^{3/2} \\ y^{1/2} \end{pmatrix}.$$

2. Suppose that the roles of x and z are reversed in the previous exercise. That is, suppose $x' = -x$ and $z' = -3z$. Describe the image of $h(x, y)$ in $\mathcal{R}_2$ in this case.
3. For the Poincaré map $\Phi(x, y) = (f(x, y), g(y))$ for the model attractor, use the results of Exercise 1 to show that $g'(y) \to \infty$ as $y \to 0$.
4. Show that it is possible to verify the transitivity condition for the Lorenz model with a solution that actually lies in the attractor.
5. Prove that arbitrarily close to any point in the model Lorenz attractor, there is a solution that eventually tends to the equilibrium point at $(0, 0, 0)$.
6. Prove that there is a periodic solution γ of the model Lorenz system that meets the rectangle R in precisely two distinct points.
7. Prove that arbitrarily close to any point in the model Lorenz attractor, there is a solution that eventually tends to the periodic solution γ from the previous exercise.